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Modeling of Borehole Modes Using the Spectral Method

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SUMMARY

We present algorithm and code that solves the dispersion equation for cylindrically layered media consisting of arbitrary number of solid elastic and fluid layers. The algorithm is based on the spectral method which discretises the underlying wave equations with the help of spectral differentiation matrices and solves the corresponding equations as an generalized eigenvalue problem. For a given frequency the eigenvalues correspond to the wavenumbers of different modes. The advantage of this technique is that it is easy to implement especially for cases where traditional root-finding methods are strongly limited or hard to realize, i.e. for attenuative, anisotropic and poroelastic media. We illustrate the application of the new approach using models of a free solid bar and a fluid-filled cylinder. These dispersion curves are in good agreement with analytical results, which confirms the accuracy of the new method.

Introduction

Modelling different wave modes propagating along a cylindrical borehole is important for understanding and quantitative interpretation of borehole sonic and seismic measurements. All these modes are strongly frequency dependent. Traditionally, mode dispersion was studied by finding roots of analytical dispersion equations. This method has a long history. By the end of the 19th century Ludwig Pochhammer and Charles Chree (e.g. Pochhammer, 1876) had already independently investigated the wave propagation along an elastic cylindrical bar. For the case of a fluid-filled borehole, with appropriate boundary conditions, analytical solutions were given by Biot (1952) and Del Grosso and McGill (1968). The case of a hollow cylinder either empty or filled with a fluid for different tube wall thicknesses, was studied e.g. by Gazis (1959) and Rubinow and Keller (1971).

The root finding technique is a direct analytical and hence the most natural method for analysis of the dispersion. However this method becomes difficult to implement when the numbers of cylindrical layers and modes become large and when inelastic effects need to be taken into account, as separation of different roots becomes challenging.

An alternative approach to modelling wave propagation in circular structures was recently introduced by Adamou and Craster (2004) based on spectral methods. The problem is solved by numerical interpolation using spectral differentiation matrices (DMs). The advantage of this approach is that it is much faster and easier to implement then conventional root-finding methods, especially for attenuative, poroelastic or anisotropic structures.

In this paper we introduce the spectral method approach for longitudinal wave propagation in circular cylindrical structures, and compare the results with the known analytical solutions for two simple cases: a free solid bar and a fluid-filled hollow cylinder.

Methodology

We introduce the spectral method using the easiest case of longitudinal wave propagation in a free solid bar. Figure 1 displays the geometry and the displacement field. We use cylindrical co-ordinates (r,θ,z). As longitudinal (axysimmetric) wave propagation in a cylinder is independent of θ, the particle motion occurs solely in the $r-z$ plane where the displacement u_r is parallel to the r-axis and u_z to the z-axis. The bar is a homogeneous, isotropic, elastic body with P- and S-wave velocity (v_p, v_s) and density ρ.

The equations describing such a system are known as the Pochhammer-Chree-equations

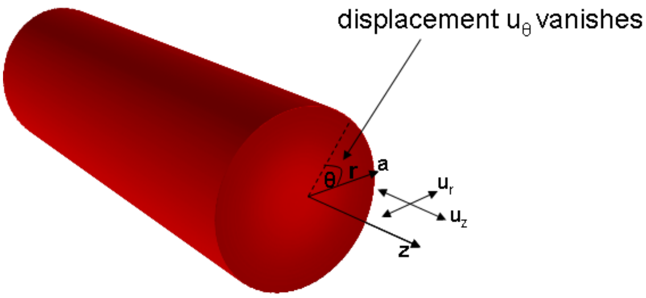

Figure 1: Geometry of a free solid bar, displaying the coordinate system which reduces to (r,z) and the displacement field (u_r,u_z) for axisymmetric wave propagation

(Pochhammer, 1876) which are presented in detail by Kolsky (1963) and Bancroft (1941). The equations of motion in polar coordinates are

$$\underbrace{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\omega^2}{v_p^2}\right)}_{\mathfrak{L}_{v_p}}\Delta = k_z^2\Delta \quad (1) \quad , \quad \underbrace{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\omega^2}{v_s^2}\right)}_{\mathfrak{L}_{v_s}}\bar{\omega}_\theta = k_z^2\bar{\omega}_\theta \quad (2) \quad ,$$

where the field variables are, Δ, which is the dilatation and $\bar{\omega}_\theta$, a component of the rotation, both in cylindrical coordinates. ω is the angular frequency and k_z the axial wavenumber. Stress-free boundary conditions are assumed at $r = a$ which means $\sigma_{rr}|_{r=a} = \sigma_{rz}|_{r=a} = 0$, where σ_{rr} is the normal stress in radial direction and σ_{rz} is the radial shear stress acting in z direction.
These abovementioned authors describe the classical approach of root-finding which yields the frequency equation $\det M(\omega, k_z) = 0$. The roots of this equation yield the dispersion relation $\omega(k_z)$. Since wave solutions in cylindrical coordinates contain various Bessel functions, it is often quite difficult to find and separate various roots. This gets more complicated in the case of leaky modes or lossy structures where solutions of the dispersion relation should be found in the complex plane.
The spectral method bypasses these difficulties and solves the underlying Helmholtz equations numerically. For elastic wave propagation this was first implemented by Adamou and Craster (2004) who investigated circumferential waves in an elastic annulus (motion independent of r and z see Fig. 1). In this study we extend the spectral method to axisymmetric longitudinal models.
The initial idea by Adamou and Craster (2004) is to use spectral DM's to discretise the wave equations and boundary conditions. Then they can be solved as a matrix eigenvalue problem. The resulting eigenvalues correspond to a wavenumber k_z for a given frequency ω or vice versa. In the following section we illustrate the process of discretisation for the case of the free solid bar using equation (1). Extension to the case of arbitrary n-layered fluid-solid media is straightforward.

Differentiation matrices

In order to solve the Helmholtz equation (1) numerically we use DM's to represent the differential operator $\mathfrak{L}_{v_p}$. Consider a function $f(x)$ evaluated at N interpolation points, which is represented in a vector $\mathbf{f}$ of length N. This interpolated function $\mathbf{f}$ is connected to its mth derivative $\mathbf{f}^{(m)}$ through the following equation

$$\begin{pmatrix} f_1^{(m)} \\ f_2^{(m)} \\ \vdots \\ f_N^{(m)} \end{pmatrix} \approx \underbrace{\begin{pmatrix} D_{11}^{(m)} & D_{12}^{(m)} & \cdots & D_{1N}^{(m)} \\ D_{21}^{(m)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ D_{N1}^{(m)} & \cdots & \cdots & D_{NN}^{(m)} \end{pmatrix}}_{D^{(m)}} \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} \quad . \tag{3}$$

This means that an approximation of the m-th derivative of $\mathbf{f}$ can be calculated by multiplying $\mathbf{f}$ with the $N \times N$ matrix $D^{(m)}$, which represents the DM. The DM's are calculated by using Chebyshev polynomials. The N interpolation points, which are, in our case, along the radius r of the cylinder, are the N roots of the Chebyshev polynomial of the N-th order. The Chebyshev DM's are calculated using the recursive formula for the derivatives of Chebyshev polynomials. The advantage of this approach is that the the derivatives of the polynomials can be computed exactly.
The interpolated $\mathbf{r}$ vector and the calculated DM's are now used to represent the differential operator $\mathfrak{L}_{vp}$ in form of a $N \times N$ matrix

$$L_{v_p} = D^{(2)} + \text{diag}\left(\frac{1}{r}\right) D^{(1)} + \text{diag}\left(\frac{\omega^2}{v_p^2}\right) \quad . \tag{4}$$

In the same way matrix representations for all equations of motion and boundary conditions are constructed.

Formulation of the eigenvalue problem

In order to solve the now numerically interpolated equations as an eigenvalue problem they have to be combined in one matrix equation. First the equations of motion L_{v_p} and L_{v_s} are combined in the $2N \times 2N$ matrix

$$P = \begin{pmatrix} L_{v_p} & 0 \\ 0 & L_{v_s} \end{pmatrix} \quad . \tag{5}$$

The stress components σ_{rr} and σ_{rz} are grouped in a matrix of the same size

$$S = \begin{pmatrix} \sigma_{rr}^{\Delta} & \sigma_{rr}^{\bar{\omega}_\theta} \\ \sigma_{rz}^{\Delta} & \sigma_{rz}^{\bar{\omega}_\theta} \end{pmatrix} \quad , \tag{6}$$

where each component is separated in terms of the field variables Δ and $\bar{\omega}_\theta$.

The last step is to combine the boundary conditions with the equations of motion in an appropriate way. As the problem is solved for a hollow cylinder with a very small inner radius which is a limiting case for a solid cylinder, we have to consider inner and outer boundary conditions. This means that the elements of S representing the interpolation points of the inner and outer boundary $(1, N, N+1$ and $2N)$ replace the corresponding rows in the P matrix which is now referred to as $\tilde{P}$. The eigenvalue problem can now be formulated in the form

$$\tilde{P}\mathbf{u} = k_z^2 Q\mathbf{u} \quad , \tag{7}$$

where the stress-free boundary conditions are set inside the matrix Q. This is a generalized eigenvalue problem and can be solved using the MATLAB routine $eig(\tilde{P}, Q)$ for instance.

This approach can be straightforwardly extended to n cylindrical fluid and solid layers. All matrices have to be computed for the properties of the certain layers and combined in a now bigger matrix P. In the case of welded boundaries the displacements have to be continuous and these have to be computed and applied to the appropriate rows of matrix $\tilde{P}$.

Dispersion curves for a solid bar and a fluid-filled cylinder

Let us illustrate the results produced by this approach in the form of dispersion curves (Fig. 2). To compare with previous results obtained by root-finding techniques, we used models presented by Kolsky (1963) and Del Grosso and McGill (1968). In Fig. 2a the dispersion curves for a free solid bar are computed with the parameters shown in the picture. These curves reproduce precisely the dispersion curves shown in Kolsky (1963, Fig.14) which are calculated analytically using root-finding techniques. The second example (Fig. 2b) is the simplest two-layer model: a fluid-filled hollow cylinder. The dispersion curves were originally calculated by Del Grosso and McGill (1968). Again we were able to reproduce these results accurately using the spectral method. Note that in this case there exist two fundamental modes starting from a zero frequency: first one (ET0) is commonly referred to as a tube wave or Stoneley wave, whereas second (ET1) is an analog of a (longitudinal) plate wave. The mode ET1 only weakly depends on the fluid properties and disappears when the thickness of the cylinder wall increases to infinity or the outer boundary of the cylinder becomes rigid (Rn).

Conclusions

We extended and implemented the spectral method for propagation of axisymmetric longitudinal modes in a cylindrical bar. The method was also generalized to N-layered cylindrical fluid-solid structures. The advantage of this approach is, that in contrast to traditional methods, it is easier to implement, especially for cases where root-finding becomes complicated. For cylindrical geometries the spectral method is a good alternative as the produced results are accurate and the computational time is very short. The method is well-suited for extension to anisotropic, attenuative and poroelastic structures.

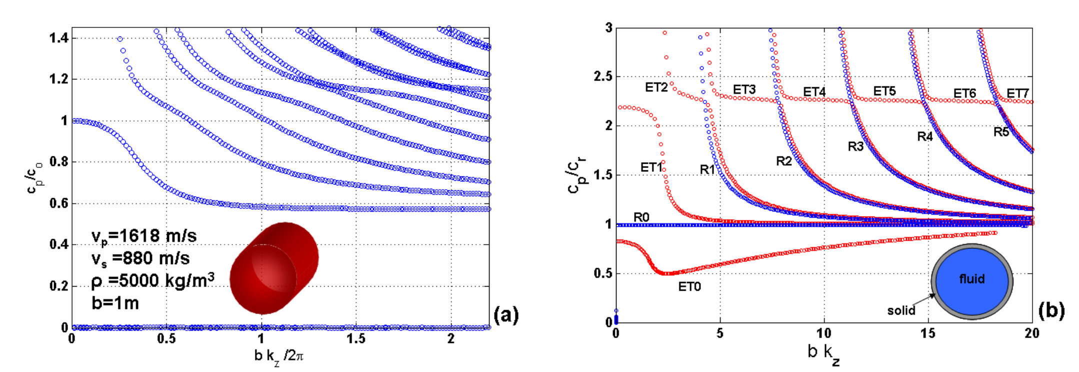

Figure 2: a) Dispersion curves of a free solid bar. x-axis: wavenumber-radius product, y-axis: phase velocity $c_p = \omega/k_z$ normalized by the bar velocity $c_0^2 = E/\rho$ where E is the Young's modulus (compare with Kolsky, 1963, Chap. 3 pp.54); b) Dispersion curves for a hollow cylinder filled with non-viscous fluid. Thickness of the cylinder wall: $1/8m$; velocities: $v_p = 3765m/s, v_s = 2012m/s$; density $8500kg/m^3$; Modes ETn in elastic tube with stress-free outer boundary are shown in red, whereas mode Rn for pipe with rigid outer boundary are shown in blue. Phase velocity c_p is normalized by Rayleigh velocity (c_r) (compare with Del Grosso and McGill, 1968)

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Acknowledgements

We are grateful to Prof. Boris Kashtan (St. Petersburg State University) who suggested the idea of applying the spectral method to the problem at hand. We thank Shell International Exploration and Production for support of this work. We would also like to thank Prof. Richard Craster (Imperial College London) for his helpful advise at the initial stage of the project.

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