Computing borehole modes with spectral method

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SUMMARY

We present algorithm and code that solves dispersion equation for cylindrically layered media consisting of arbitrary number of solid elastic and fluid layers. The algorithm is based on the spectral method which discretises the underlying wave equations with the help of spectral differentiation matrices and solves the corresponding equations as an generalized eigenvalue problem. For a given frequency the eigenvalues correspond to the wavenumbers of different modes. The advantage of this technique is that it is easy to implement especially for cases where traditional *root-finding* methods are strongly limited or hard to realize, i.e. for attenuative, anisotropic and poroelastic media. We illustrate the application of the new approach using models of a free solid bar and a fluid-filled cylinder. The computed dispersion curves are in good agreement with analytical results, which confirms the accuracy of the new method.

INTRODUCTION

Modelling different wave modes propagating along a cylindrical borehole is important for understanding and quantitative interpretation of borehole sonic and seismic measurements. All these modes are strongly frequency dependent. Traditionally, mode dispersion was studied by finding roots of analytical dispersion equations. This method has a long history. At the end of the 19th century Ludwig Pochhammer and Charles Chree (e.g. Pochhammer, 1876) independently investigated the wave propagation along an elastic cylindrical bar. The dispersion curves for a free cylinder were computed much later by Bancroft (1941) and Davies (1948). For the case of a fluid-filled borehole, with appropriate boundary conditions, analytical solutions were given by Biot (1952) and Del Grosso and McGill (1968). The case of a hollow cylinder either empty or filled with a fluid for different tube wall thicknesses, was studied e.g. by Gazis (1959) and Rubinow and Keller (1971).

Root-finding is a direct analytical technique and hence the most natural method for analysis of the dispersion. However this method becomes difficult to implement when the numbers of cylindrical layers and modes become large and when inelastic effects need to be taken into account, as separation of different roots becomes challenging.

An alternative approach to modelling wave propagation in circular structures was recently introduced by Adamou and Craster (2004) based on spectral methods. The problem is solved by numerical interpolation using spectral differentiation matrices (DMs). The advantage of this approach is that it is much faster and easier to implement then conventional *root-finding* methods, especially for attenuative, poroelastic or anisotropic structures.

In this paper we introduce the spectral method approach for longitudinal wave propagation along the axis of a free circular cylinder. The results are compared with the known analytical solutions from Davies (1948). The approach is then extended to n cylindrical solid and fluid layers. We illustrate the results with the model of a fluid filled tube, which are compared to the results from A. Sidorov (St. Petersburg, State University, Russia) *root-finding* program based on the parameters used by Del Grosso and McGill (1968). Finally particle displacement profiles,which result from the eigenvectors, are computed. The results for the free cylinder are illustrated and discussed for various frequencies.

THE UNDERLYING EQUATIONS

We introduce the spectral method using the easiest case of longitudinal wave propagation in a free solid bar. Fig. 1 displays the geometry and the displacement field. We use cylindrical coordinates (r,θ,z). As longitudinal (axisymmetric) wave propagation in a cylinder is independent of θ, the particle motion occurs solely in the $r-z$ plane where the displacement u_r is parallel to the r-axis and u_z to the z-axis. We consider the propagation of a infinite train of sinusoidal waves along the z-axis of the cylinder which is a harmonic function of z and t of the form

$$\begin{aligned} u_r &= Ue^{i(k_z z+\omega t)} \quad , \\ u_z &= We^{i(k_z z+\omega t)} \quad , \end{aligned} \tag{1}$$

where ω is the angular frequency and k_z the angular axial wavenumber. U and W are the amplitudes which are functions of r and θ. From eqs. 1 it follows that $\partial u_r/\partial t = i\omega u_r$ and $\partial u_z/\partial z = ik_z u_z$ etc. The bar is a homogeneous, isotropic, elastic body with P- and S-wave velocity (v_p, v_s) and density ρ.

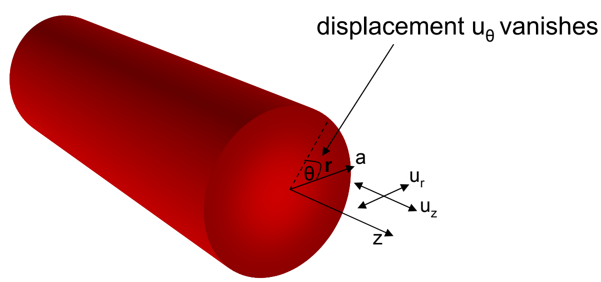

Figure 1: Geometry of a free solid bar, displaying the coordinate system which reduces to (r,z) and the displacement field (u_r,u_z) for axisymmetric wave propagation.

The equations describing such a system are known as the Pochhammer-Chree-equations (Pochhammer, 1876) which are presented in detail by Kolsky (1963) and Bancroft (1941). The equations of motion in polar coordinates using displacement potentials are

$$\underbrace{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\omega^2}{v_p^2}\right)}_{\mathfrak{L}_{v_p}}\phi = k_z^2\phi \quad , \tag{2}$$

$$\underbrace{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\omega^2}{v_s^2}\right)}_{\mathfrak{L}_{v_p}}\psi_\theta = k_z^2\psi_\theta \quad , \tag{3}$$

where the scalar potential is ϕ and ψ_θ is the θ component of the vectorial potential. The stress-strain relations for axial-symmetric modes are

$$\sigma_{rr} = \lambda\Delta + 2G\frac{\partial u_r}{\partial r} \quad , \tag{4}$$

$$\sigma_{rz} = G\left(ik_z u_r + \frac{\partial u_z}{\partial r}\right) \quad , \tag{5}$$

where Δ is the dilatation in cylindrical $r-z$ coordinates. λ and G are the Lame parameter. Stress-free boundary conditions are assumed at $r=a$ which means $\sigma_{rr}|_{r=a} = \sigma_{rz}|_{r=a} = 0$. σ_{rr} is the normal stress in radial direction and σ_{rz} is the radial shear stress acting in z direction.

In order to complete the set of equations the displacements u_r in r-direction and u_z in z-direction are defined in $\omega - k_z$ domain as

$$u_r = -\frac{\partial \phi}{\partial r} - ik_z\psi_\theta \quad , \tag{6}$$

$$u_z = ik_z\phi + \frac{1}{r}\frac{\partial(r\psi_\theta)}{\partial r} \quad . \tag{7}$$

These equations fully describe the problem of any free vibrating cylindrical structures in the $r-z$ plane. In the next section a new approach, based on the spectral method, is introduced in order to solve these equations.

METHODOLOGY

The *root-finding* approach was developed by Pochhammer (1876) and Chree (1889) in the 19th century. A general solution to eqs. (2)-(3) is found which is a combination of Bessel functions of different order. Substituting the solution into the boundary conditions yields a homogenous system of linear algebraic equations. In order to have non-trivial solutions the determinant of its matrix M must be equal to zero, $\det M(\omega, k_z) = 0$. This is called the frequency equation. The roots of this equation yield the dispersion relation $\omega(k_z)$. Since wave solutions in cylindrical coordinates contain various Bessel functions, it is often quite difficult to find and separate various roots. This gets more complicated in the case of leaky modes or lossy structures where solutions of the dispersion relation should be found in the complex plane.

The spectral method bypasses these difficulties and solves the underlying Helmholtz equations numerically. For elastic wave propagation this was first implemented by Adamou and Craster (2004) who investigated circumferential waves in an elastic annulus (motion independent of r and z: see Fig. 2). In this study we extend the spectral method to axisymmetric longitudinal models.

The initial idea by Adamou and Craster (2004) is to use spectral DM's to discretise the wave equations and boundary conditions. Then they can be solved as a matrix eigenvalue problem. The resulting eigenvalues correspond to a wavenumber k_z for a given frequency ω or vice versa. In the following section we illustrate the process of discretisation for the case of the free solid bar using eq. (2).

Subsequently the method is straightforwardly extended to the case of arbitrary n-layered fluid-solid media. The eigenvectors correspond to the potentials ϕ and ψ_θ which are finally used to compute the mode-shapes.

DIFFERENTIATION MATRICES

In order to solve the Helmholtz eq. (2) numerically we use DM's to represent the differential operator L_{v_p}. Consider a function $f(x)$ evaluated at N interpolation points, which is represented in a vector $\mathbf{f}$ of length N. This interpolated function $\mathbf{f}$ is connected to its mth derivative $\mathbf{f}^{(m)}$ through the following equation

$$\begin{pmatrix} f_1^{(m)} \\ f_2^{(m)} \\ \vdots \\ f_N^{(m)} \end{pmatrix} \approx \underbrace{\begin{pmatrix} D_{11}^{(m)} & D_{12}^{(m)} & \cdots & D_{1N}^{(m)} \\ D_{21}^{(m)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ D_{N1}^{(m)} & \cdots & \cdots & D_{NN}^{(m)} \end{pmatrix}}_{D^{(m)}} \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} \quad . \tag{8}$$

This means that an approximation of the m-th derivative of $\mathbf{f}$ can be calculated by multiplying $\mathbf{f}$ with the $N \times N$ matrix $D^{(m)}$, which represents the DM. The DM's are calculated by using Chebyshev polynomials. The N interpolation points, which are, in our case, along the radius r of the cylinder, are the N roots of the Chebyshev polynomial of the N-th order. The Chebyshev DM's are calculated using the recursive formula for the derivatives of Chebyshev polynomials. The advantage of this approach is that the derivatives of the polynomials can be computed exactly.

The interpolated $\mathbf{r}$ vector and the calculated DM's are now used to represent the differential operator $\mathfrak{L}_{vp}$ (eq. 2) in form of a $N \times N$ matrix

$$L_{v_p} = D^{(2)} + \mathrm{diag}\left(\frac{1}{r}\right)D^{(1)} + \mathrm{diag}\left(\frac{\omega^2}{v_p^2}\right) \quad . \tag{9}$$

In the same way matrix representations for all equations of motion and boundary conditions are constructed.

FORMULATION OF THE EIGENVALUE PROBLEM

In order to solve the now numerically interpolated equations as an eigenvalue problem they have to be combined in one matrix equation. First the equations of motion L_{v_p} and L_{v_s} are combined in the $2N \times 2N$ matrix

$$P = \begin{pmatrix} L_{v_p} & 0 \\ 0 & L_{v_s} \end{pmatrix} \quad . \tag{10}$$

The stress components σ_{rr} and σ_{rz} are grouped in a matrix of the same size

$$S = \begin{pmatrix} \sigma_{rr}^{\phi} & \sigma_{rr}^{\psi_\theta} \, , \\ \sigma_{rz}^{\phi} & \sigma_{rz}^{\psi_\theta} \end{pmatrix} \quad , \tag{11}$$

where each component is separated in terms of the displacement potentials ϕ and ψ_θ.

The last step is to combine the boundary conditions with the equations of motion in an appropriate way. As the problem is solved for a hollow cylinder with a very small inner radius which is a limiting case for a solid cylinder, we have to consider inner and outer boundary conditions. This means that the elements of S representing the interpolation points of the inner and outer boundary $(1, N, N+1$ and $2N)$ replace the corresponding rows in the P matrix which is now referred to as $\tilde{P}$. The eigenvalue problem can now be formulated in the form

$$\tilde{P}\mathbf{u} = k_z^2 Q\mathbf{u} \quad , \tag{12}$$

where the stress-free boundary conditions are set inside the matrix Q. This is a generalized eigenvalue problem and can be solved,for instance using the MATLAB routine $eig(\tilde{P}, Q)$.

CYLINDRICAL LAYERING

This approach can be straightforwardly extended to n cylindrical fluid and solid layers (see Fig. 2). Each of the n layers has P- and S-wave velocities $(v_{p_1}...v_{p_n}, v_{s_1}...v_{s_n})$ and densities $\rho_1...\rho_n$. For each of these layers the matrix P_n is computed in analogy to eq. (10). These equations are finally combined in a diagonal matrix of the size $n2N \times n2N$ which has the form

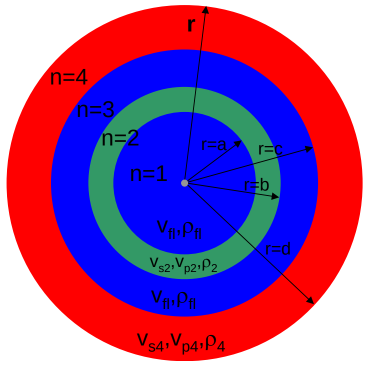

Figure 2: Geometry of a model with four cylindrical layers. The layer index is $n = 1..4$ numbered from the center to the surface of the bar; The layers are either non-viscous fluid $(v_{fl},\ \rho_{fl})$ or elastic solid $(v_{pn},\ v_{sn},\ \rho_{fl})$

$$P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_n \end{pmatrix} . \tag{13}$$

The same procedure has to be done for the stress components S_n (see eq. 11) of each layer n, which are finally combined in a matrix S of same size as P. A similar matrix U is computed for the displacement components.

For the case of layering additional boundary conditions across the interface have to be introduced. Both the stress and the displacement components have to be continuous. This means that $[\sigma_{rr}]\Big|_{r=i} = [\sigma_{rz}]\Big|_{r=i} = [u_r]\Big|_{r=i} = [u_z]\Big|_{r=i} = 0$, where i are the radii of the different layers $i = a, b....$ The stress-free boundary conditions on the inner and outer boundary are introduced in the P-matrix the same way as for a free cylinder in the rows $(1, nN, nN+1$ and $n2N)$. The interface conditions are introduced as the vanishing differences of the displacements and stresses between adjacent layers. It is convenient to apply the continuity stress boundary conditions to the rows $N, 2N...nN$ of the P-matrix and the continuity displacement conditions to $N, 2N...nN$.

This means that the elements of S and U representing the interpolation points of the inner and outer boundary and the interfaces replace the corresponding rows in the P matrix which is now referred to as $\tilde{P}$. The eigenvalue problem can now be formulated analogous to eq. 12 and solved using the MATLAB eigenvalue routine.

DISPERSION CURVES FOR A SOLID BAR AND A FLUID-FILLED CYLINDER

Let us illustrate the results produced by this approach in the form of dispersion curves (Fig. 3-4). To compare with previous results obtained by root-finding techniques, we used models presented by Davies (1948) and Del Grosso and McGill (1968). In Fig. (3) the dispersion curves for a free solid bar are computed with the parameters shown in the picture. These curves reproduce in very good agreement the dispersion curves shown in Davies (1948,Fig.13) which are calculated analytically using root-finding techniques. The fundamental mode $L(0,1)$ behaves like a pure extensional mode for low frequencies and propagates with the velocity $\sqrt{E/\rho}$ where E is the Young's modulus. For higher frequencies the mode propagates like a Rayleigh wave on the cylinder surface. The higher modes $(L(0,1)...L(0,n))$ have cut-off frequencies, which means they don't exist below these frequencies. For very high frequencies they tend to propagate close to the Rayleigh velocity.

The second example (Fig. 4) is a two-layer model: a fluid-filled hollow cylinder. The dispersion curves were originally calculated by Del Grosso and McGill (1968). Here the dispersion curves were computed by A. Sidorov using the *root-finding* technique analogous to Del Grosso and McGill (1968). Again we were able to reproduce these results accurately using the spectral method. Note that in this case there exist two fundamental modes starting from a zero frequency: first one (ET0) is commonly referred to as a tube wave or Stoneley wave, whereas second (ET1) is an analog of a (longitudinal) plate wave. The mode ET1 only weakly depends on the fluid properties and disappears when the thickness of the cylinder wall increases to infinity or the outer boundary of the cylinder becomes rigid (Rn).

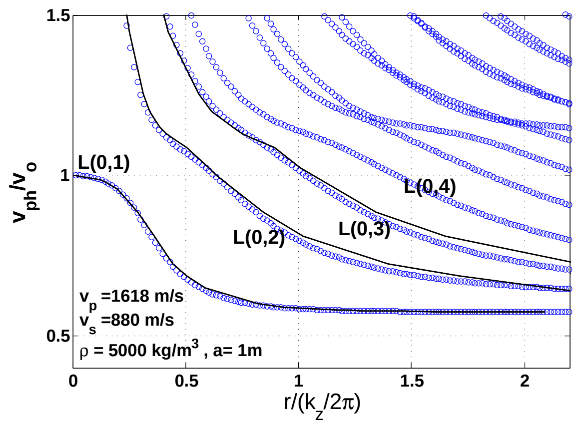

Figure 3: Dispersion curves of a free solid bar: x-axis: wavenumber-radius product, y-axis: phase velocity $v_{ph} = \omega/k_z$ normalized by the bar velocity $v_0^2 = E/\rho$ where E is the Young's modulus (compare with Davies, 1948,Sec. 11, Fig.13);

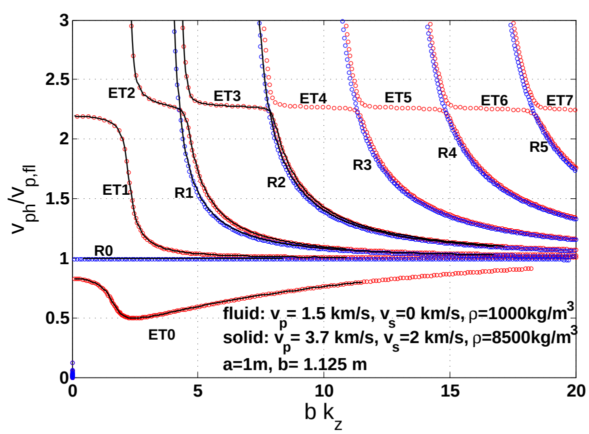

Figure 4: Dispersion curves for a hollow cylinder filled with non-viscous fluid. Thickness of the cylinder wall: $0.125m$; Modes ETn in elastic tube with stress-free outer boundary are shown in red, whereas mode Rn for pipe with rigid outer boundary are shown in blue. Phase velocity v_{ph} is normalized by the velocity of the fluid $(v_{p,fl})$ compare with Del Grosso and McGill (1968).

MODESHAPES: PARTICLE DISPLACEMENT PROFILES

Solving the eigenvalue problem yields the eigenvalues which allow to construct the dispersion curves. At the same time the eigenvectors are computed representing the potentials ϕ and ψ_θ. They allow the

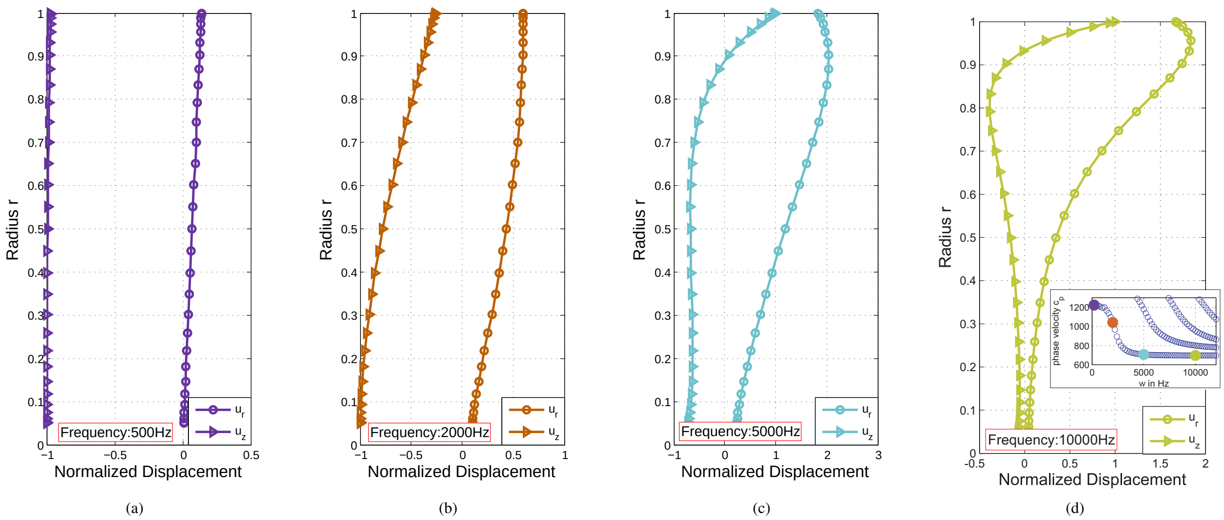

Figure 5: Particle displacement profiles of the fundamental longitudinal mode $L(0,1)$ for (a) 500 Hz (b) 2000 Hz,(c) 5000 Hz,(d) 10000 Hz; x-axis: normalized $u_r = |u_r|$ and $u_z = i|u_z|$ displacement; y-axis: bar radius from 0 m (center of bar) to 1 m (surface of bar).

computation of the modeshapes, that is the distribution of field quantities like displacements, stresses, power flow etc., along the radius of the cylinder. Exemplarily we illustrate the displacements (u_r, u_z) here which can be easily computed using the eigenvectors and eqs. (4)-(5). In order to display the particle displacement profiles u_r and u_z are calculated along the radius for a certain frequency. These values are normalized by the maximum absolute value of the u_z displacement. Finally the radial displacement is plotted as $u_r = |u_r|$ and the longitudinal displacement as $u_z = i|u_z|$.

DISPLACEMENT PROFILES OF THE $L(0,1)$ MODE

For the illustration of the displacement profiles we have chosen the fundamental mode $L(0,1)$ propagating in a free solid cylinder (see Fig. 1). The particle motion u_r and u_z is computed for four different frequencies (500 Hz, 2000 Hz, 5000 Hz and 10000 Hz). The insert plot in Fig. 5:d shows the position of the frequencies on the dispersion curve. Fig. 5:a-d displays the displacement profiles for u_r and u_z for the different frequencies.

For low frequencies like 500 Hz (Fig. 5:a) the wave propagates like a longitudinal wave. Consequently the particle motion is in axial direction mainly and uniform throughout the radius of the cylinder. The radial displacement is very small.

In Fig. 5:b we can see that for 2000 Hz the u_r displacement has already significantly increased all over the cross section. It only remains zero in the center of the cylinder. At the same time the u_z displacement decreases but keeps its maximum value in the center.

For a higher frequency (5000 Hz; Fig. 5:c) it can be observed that the shape of the displacement profiles propagates slowly towards the typical pattern of Rayleigh modes. Close to the surface (r=$0.85-1$m) the motion is already Rayleigh-like. Only towards the center of the bar especially the u_z component is still relatively strong.

Finally in Fig. 5:d we get the typical particle motion profile of Rayleigh waves. In contrast to Fig. 5:c obviously the amplitudes of both displacement components decreases significantly for $r < 0.8m$.

Summarizing we can say that the displacement field for the $L(0,1)$ mode can be modeled using the spectral method like expected, which is another proof that the approach works properly.

CONCLUSIONS

We extended and implemented the spectral method for propagation of axisymmetric longitudinal modes in a cylindrical bar. The method was also generalized to N-layered cylindrical fluid-solid structures typical for borehole environment. Dispersion curves for a free solid cylinder and a fluid filled tube were computed and compared with analytical solutions. Furthermore the displacement profile for the $L(0,1)$ were computed using the eigenvectors and displayed for different frequencies. The advantage of this approach is, that in contrast to traditional methods, it is easier to implement, especially for cases where root-finding becomes complicated. For cylindrical geometries the spectral method is a good alternative as the produced results are accurate and the computational time is very short. The method is well-suited for extension to anisotropic, attenuative and poroelastic borehole structures.

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EDITED REFERENCES

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