Seismic upscaling of layered anisotropic media

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Summary

All measurements of anisotropic coefficients in exploration seismology are made over finite volumes of inevitably heterogeneous rocks and, therefore, represent some effective values. As a result, a geophysicist often needs to estimate how unaccounted heterogeneity influences (or biases) the estimates of anisotropy. We show that this influence is usually weaker than one might think.

In particular, we demonstrate that the difference between any effective quantity $\mathbf{m}^e$ and its arithmetic average $\bar{\mathbf{m}}$ over a given volume is always *quadratic* in the local fluctuations $\tilde{\mathbf{m}}$, i.e., $\mathbf{m}^e - \bar{\mathbf{m}} = O(\tilde{\mathbf{m}}^2)$. We prove this statement under quite general conditions that have mathematical nature rather than express specifics of a given quantity $\mathbf{m}$. To verify our theoretical findings, we perform Backus- and Dix-type averages of a typical well log intentionally "contaminated" by moderate anisotropy.

Introduction

In-situ seismic anisotropy can be estimated using a variety of techniques. Since they require data that are typically collected over some finite intervals of heterogeneous earth, one usually needs to separate the influence of anisotropy on the measurements from that of heterogeneity. It is not an easy task because, depending on the frequency of propagating waves, heterogeneity and anisotropy might mimic each other. The well known example of this kind is a finely layered isotropic solid that effectively behaves as a vertically transversely isotropic (VTI) rock when probed by sufficiently long seismic waves (Backus, 1962). Similar transitions from heterogeneity to anisotropy and back occur over the whole range of seismic frequencies. For instance, a layered isotropic structure also looks like a VTI one at high frequencies, when the ray tracing and Dix-type averaging of the normal moveout (NMO) velocities can be applied.

In our view, part of the difficulty in discriminating the influences of intrinsic anisotropy and heterogeneity on seismic data stems from the inherent complexity of wave propagation in heterogeneous anisotropic media. Even restricting the models to 1-D horizontally layered media does not fully eliminate the problem. Indeed, the effective stiffnesses $\mathbf{c}^{\,e}$ in the low-frequency regime are computed by performing Schoenberg-Muir (1989) calculus that operates with elements of the interval stiffness tensors $\mathbf{c}$ arranged in 3×3 matrices. At high frequencies, the anisotropic counterpart of conventional Dix (1955) formula averages the NMO ellipses represented by 2×2 matrices $\mathbf{W}$ (Grechka et al., 1999). Given such a complexity and usual inaccuracies in the estimated anisotropy that preclude a geophysicist from choosing the best possible model, it is not surprising that physical intuition regarding to interplay between anisotropy and heterogeneity has not been developed yet.

Our goal is to show that this interdependence is often not strong and, therefore, can be ignored. Specifically, we will demonstrate that all averaging techniques are equivalent up to the terms *quadratic* in fluctuations $\tilde{\mathbf{m}}$ of layer velocities, densities, and anisotropies from their mean (i.e., volume average) values $\bar{\mathbf{m}}$. As a consequence, the errors made by replacing the properly computed effective anisotropic coefficients $\mathbf{\Delta}^e$ with their means $\bar{\mathbf{\Delta}}$ are normally expected to be small. We prove this statement and verify it using a typical well log with intentionally added moderate anisotropy.

Local and effective quantities

In this section, we discuss the general relationship between a pair of interval (or local) and effective quantities, $\mathbf{m}$ and $\mathbf{m}^e$. Quite remarkably, all known averaging procedures applied to either tensors, or matrices, or scalars $\mathbf{m}$ and $\mathbf{m}^e$ can be represented in the form

$$\mathbf{F}\,[\,\mathbf{m}^e\,] = \frac{1}{V} \int\limits_V \mathbf{F}\,[\,\mathbf{m}(\mathbf{x})\,]\,d\mathbf{x}\,, \tag{1}$$

where $\mathbf{F}$ is the operation that averages the quantity $\mathbf{m}(\mathbf{x})$ over the volume V (which can be also an area or a linear segment), and $\mathbf{x} \equiv [x_1,\, x_2,\, x_3]$ denotes cartesian coordinates.

Although the functions $\mathbf{F}$ are different for different averaging procedures (see the examples below), they share two important properties: differentiability and invertibility. The first property means the existence of derivatives $D\mathbf{F}/D\mathbf{m} \equiv \partial F_i/\partial m_j$, where F_i and m_j denote the elements of tensors $\mathbf{F}$ and $\mathbf{m}$. The invertibility states that there is such a function $\mathbf{F}^{\,\dagger}$ which undoes $\mathbf{F}$, i.e.,

$$\mathbf{F}^{\,\dagger}\,[\,\mathbf{F}\,[\,\mathbf{m}\,]\,] = \mathbf{m} \quad \text{and} \quad \mathbf{F}^{\,\dagger}\,[\,\mathbf{F}\,[\,\mathbf{m}^e\,]\,] = \mathbf{m}^e\,. \tag{2}$$

The latter allows us to rewrite equation (1) as

$$\mathbf{m}^e = \mathbf{F}^{\dagger} \left[\frac{1}{V} \int\limits_V \mathbf{F}\,[\,\mathbf{m}(\mathbf{x})\,]\, d\mathbf{x} \right] . \tag{3}$$

Examples of functions F

Since any averaging procedure can be cast in the form (1) or (3), the examples of functions **F** are abundant. Perhaps, the simplest ones are Backus (1962) averages of Lamè constants μ in a stack of plane thin isotropic layers that yield the stiffness coefficients c_{66}^e and c_{44}^e of the effective VTI medium

$$c_{66}^e = \frac{1}{V} \int\limits_V \mu(\mathbf{x})\, d\mathbf{x} \quad \text{and} \quad c_{44}^e = \left[\frac{1}{V} \int\limits_V [\,\mu(\mathbf{x})]^{-1}\, d\mathbf{x} \right]^{-1} . \tag{4}$$

Clearly, equations (4) are special cases of equation (3) when both **F** and $\mathbf{F}^{\dagger}$ are the scalar functions given by $F[m] = F^{\dagger}[m] = m$ and $F[m] = F^{\dagger}[m] = m^{-1}$, respectively. A more complicated Schoenberg-Muir (1989) calculus, which extends Backus (1962) averaging to anisotropic media, can be also written in the form (3).

Other averaging procedures, for instance, Dix (1955) and generalized Dix (Grechka et al., 1999) formulae for the NMO velocities v and the NMO ellipses (2×2 matrices) **W**,

$$v^e = \left[\frac{1}{T} \int\limits_0^T v^2(t)\, dt \right]^{1/2} \quad \text{and} \quad \mathbf{W}^e = \left[\frac{1}{T} \int\limits_0^T \mathbf{W}^{-1}(t)\, dt \right]^{-1} , \tag{5}$$

can be also reduced to equations (1) or (3) that perform spatial rather than temporal integration. The way to show this is to change the differential dt in the equations above to $q\, dx_3$, where q is the vertical slowness and x_3 is the depth; T denotes the vertical time in equations (5).

Proof of relationship $\mathbf{m^e} = \bar{\mathbf{m}} + \mathbf{O}(\tilde{\mathbf{m}}^2)$

Having presented a number of examples illustrating the validity of equations (1) and (3), we are ready to formulate our main statement. First, however, we need to define the mean $\bar{\mathbf{m}}$ and fluctuation $\tilde{\mathbf{m}}(\mathbf{x})$ of $\mathbf{m}(\mathbf{x})$:

$$\bar{\mathbf{m}} = \frac{1}{V} \int\limits_V \mathbf{m}(\mathbf{x})\, d\mathbf{x} \tag{6}$$

and

$$\tilde{\mathbf{m}}(\mathbf{x}) = \mathbf{m}(\mathbf{x}) - \bar{\mathbf{m}} \,. \tag{7}$$

As follows from equations (6) and (7),

$$\int\limits_V \tilde{\mathbf{m}}(\mathbf{x})\, d\mathbf{x} = 0 \,. \tag{8}$$

The equality we intend to prove reads

$$\mathbf{m}^e = \bar{\mathbf{m}} + O(\tilde{\mathbf{m}}^2) \,. \tag{9}$$

Let us note the remarkable fact that $\mathbf{m}^e$ given by equation (9) does *not* contain any linear terms in fluctuation $\tilde{\mathbf{m}}$. To show this, we expand $\mathbf{F}(\mathbf{m})$ in Taylor series in the vicinity of $\bar{\mathbf{m}}$,

$$\mathbf{F}\,[\,\mathbf{m}\,] = \mathbf{F}\,[\,\bar{\mathbf{m}}\,] + \left. \frac{D\mathbf{F}}{D\mathbf{m}} \right|_{\mathbf{m}=\bar{\mathbf{m}}} \tilde{\mathbf{m}} + O(\tilde{\mathbf{m}}^2) \,, \tag{10}$$

and substitute this expansion into equation (3). Due to the property (8), we find

$$\mathbf{m}^e = \mathbf{F}^{\dagger} \left[\mathbf{F}\,[\,\bar{\mathbf{m}}\,] + O(\tilde{\mathbf{m}}^2) \right] . \tag{11}$$

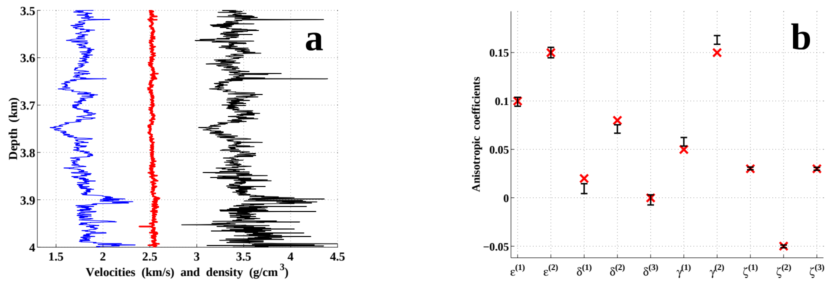

Fig. 1: (a) Sonic (black), shear (blue), and density (red) logs. We treat sonic and shear logs as the vertical velocities V_{P0} and V_{S0}, respectively. (b) Mean values of interval anisotropic coefficients (crosses) and ranges of their effective values $\mathbf{\Delta}^e$ (bars) obtained for 300 models.

Expanding function $\mathbf{F}^\dagger$ in Taylor series further and making use of the first equation (2), we obtain

$$\mathbf{m}^e = \mathbf{F}^\dagger\left[\mathbf{F}\left[\bar{\mathbf{m}}\right]\right] + \left.\frac{D\mathbf{F}^\dagger}{D\{\mathbf{F}\left[\bar{\mathbf{m}}\right] + O(\tilde{\mathbf{m}}^2)\}}\right|_{\tilde{\mathbf{m}}=0} O(\tilde{\mathbf{m}}^2) + \ O(\tilde{\mathbf{m}}^4) = \bar{\mathbf{m}} + O(\tilde{\mathbf{m}}^2) \tag{12}$$

and, thus, prove equality (9).

It is critical to realize that since no formal relationship between the norms $||\bar{\mathbf{m}}||$ and $||\tilde{\mathbf{m}}||$ was used to establish equation (9), it has little value because the relative magnitudes of its terms $\bar{\mathbf{m}}$ and $O(\tilde{\mathbf{m}}^2)$ can be arbitrary. To make our result more practical, we assume that $||\bar{\mathbf{m}}|| \gg ||\tilde{\mathbf{m}}||$. Then, the usefulness of equation (9) becomes obvious. Indeed, not only does it show that any effective quantity $\mathbf{m}^e$ is approximately equal to the mean value $\bar{\mathbf{m}}$ of the corresponding local quantity $\mathbf{m}$ but also states that the error one makes by replacing $\mathbf{m}^e$ with $\bar{\mathbf{m}}$ is expected to be small because it is *quadratic* with respect to the fluctuation $\tilde{\mathbf{m}}$. As a result, the approximation

$$\mathbf{m}^e \approx \bar{\mathbf{m}} \tag{13}$$

might be acceptable in many practical applications.

Numerical example

Here we demonstrate that approximation (13) works surprisingly well for typical layered media characterized by moderate polar and azimuthal anisotropy. We compare the performance of our approximation in predicting both the effective anisotropic coefficients $\mathbf{c}^e$ computed by Schoenberg-Muir (1989) calculus and the effective NMO ellipses $\mathbf{W}^e$ obtained using the generalized Dix formula (Grechka et al., 1999).

Figure 1a displays portions of typical Gulf of Mexico sonic, shear, and density logs measured over 500 m interval with the increment 0.5 ft = 0.1524 m. Note that vertical velocity heterogeneity is not that weak. Its values, expressed through the differences between velocity maxima and minima divided by the mean velocities, are 49% and 51% for P- and S-waves, respectively. As we will see below, such velocity fluctuations do not cause our approximation to break down because the volumes occupied by extreme low- and high-velocity layers are relatively small. Next, we make all layers anisotropic artificially introducing moderate monoclinic anisotropy specified by the coefficients $\mathbf{\Delta} \equiv \{\epsilon^{(1)}, \epsilon^{(2)}, \delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \gamma^{(1)}, \gamma^{(2)}, \zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}\}$ (Grechka et al., 2000) whose values are Gaussian random numbers. The means $\bar{\mathbf{\Delta}}$ of these anisotropic coefficients in the whole model are shown with crosses in Figure 1b.

To produce a suite of 300 anisotropic models, we keep the velocities and densities shown in Figure 1a fixed and randomly vary anisotropic coefficients in all layers. The standard deviations are equal to 0.10 for the interval $\epsilon^{(1,2)}$, $\delta^{(1,2,3)}$, and $\gamma^{(1,2)}$ and to 0.03 for $\zeta^{(1,2,3)}$. Given such large standard deviations, the local anisotropic coefficients $\epsilon^{(1,2)}$, $\delta^{(1,2,3)}$, and $\gamma^{(1,2)}$ would cover the whole range in Figure 1b if we plotted their interval values.

Assuming sufficiently low frequencies of propagating waves, we compute the exact effective medium for each of our 300 random models using Schoenberg and Muir (1989) calculus. The bars in Figure 1b indicate the ranges of found effective anisotropic coefficients $\mathbf{\Delta}^e$. Quite remarkably, their values are well predicted by the means of the corresponding interval coefficients (crosses). This suggests a simple practical recipe for obtaining any given effective anisotropic coefficient Δ_i^e. Instead of going through Schoenberg-Muir computations in their full complexity, which would require knowing all interval velocities and anisotropies, one can just calculate the mean $\bar{\Delta}_i$ and use it as a reasonable estimate of Δ_i^e. Figure 1b demonstrates, however, that some of such means, for example, $\bar{\delta}^{(1,2)}$ and $\bar{\gamma}^{(1,2)}$, might be biased. The origin of these biases can be explained based on Backus (1962) average of the original isotropic layered model (Figure 1a) that produces a weakly VTI solid with negative Thomsen anisotropic coefficient $\delta^{\,e}$ and

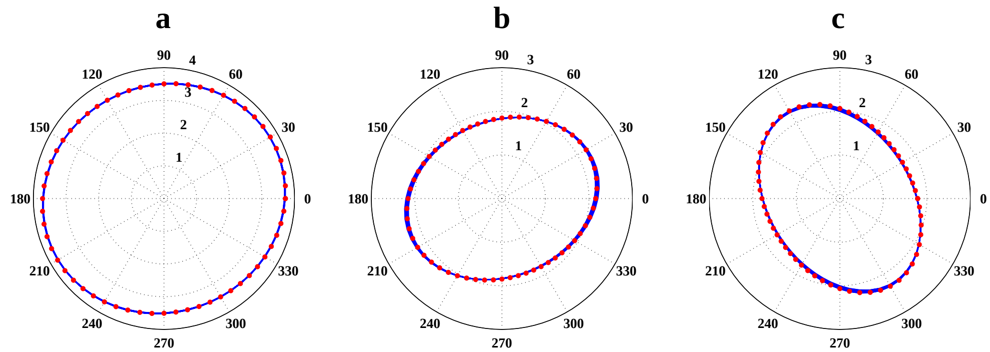

Fig. 2: NMO ellipses $\bar{\boldsymbol{W}}$ of (a) P-, (b) S_1-, and (c) S_2-waves computed in the mean model (dots) and ellipses $\boldsymbol{W}^e$ obtained by applying generalized Dix formula in 300 models with randomly varying anisotropy (solid). The circles marked with 1, 2, 3, 4 indicate velocities (in km/s), the numbers 0, 30, ..., 330 correspond to azimuths (in degrees) in the horizontal plane.

positive γ^e. The important point here is, though, that the biases are much smaller than usual errors expected in those coefficients if they were estimated from seismic data.

The effective normal-moveout velocities of high-frequency waves propagating in the generated anisotropic models are computed using the generalized Dix formula (Grechka et al., 1999). Figure 2 shows that the effective NMO ellipses $\mathbf{W}^e$ in our 300 vastly different anisotropic models overlap (thin solid lines). More importantly, the ellipses $\bar{\mathbf{W}}$ computed in the mean model (dots in Figure 2) match the exact ones. Although this could have been predicted from the correspondence of the mean and effective anisotropic coefficients shown in Figure 1b, the performed test demonstrates that robustness of formula (13) does not seem to depend on frequency content of the data.

Discussion and extensions

The main result of our study is given by the equality $\mathbf{m}^e = \bar{\mathbf{m}} + O(\tilde{\mathbf{m}}^2)$. Not only it shows that the effective parameters can be approximately replaced with mean values of the interval ones (this has been noticed before for particular models) but also indicates the high accuracy of this substitution when fluctuations of local parameters are relatively small. We demonstrated the validity of this statement for a typical well log that was intentionally made anisotropic. While it is clear that the accuracy of approximation $\mathbf{m}^e \approx \bar{\mathbf{m}}$ deteriorates as the parameter fluctuations increase, the presented test as well as others (not shown) indicate that heterogeneity should be rather strong to render our results useless.

It is interesting that equation $\mathbf{m}^e = \bar{\mathbf{m}} + O(\tilde{\mathbf{m}}^2)$ is formally valid for any effective quantity and for any frequency of propagating waves. We illustrated this by comparing the mean anisotropy $\bar{\boldsymbol{\Delta}}$ with the effective ones given by Schoenberg-Muir and generalized Dix averages which, strictly speaking, correspond to zero and infinite frequencies. A similar result for velocities and attenuation in layered isotropic structures was obtained by Shapiro and Hubral (1996). They explicitly showed that frequency-dependent phase increments and attenuation coefficients do not contain the terms linear in velocity and density fluctuations [their equations (5) and (6)]. The absence of linear terms in either Shapiro and Hubral's or our equations indicates that replacing $\bar{\mathbf{m}}$ with either harmonic or geometric or any other average will not significantly alter the predicted effective quantities.

Finally, the generality of derivation presented here also suggests that our main result remains valid not only for 1-D but also for 3-D anisotropic heterogeneous media. Indeed, since the above introduced averaging operation $\mathbf{F}$ is rather arbitrary, there is no apparent reason to restrict our final conclusion to just horizontally layered solids. While we are not aware of any exact solutions that could be used to verify the last statement for generally heterogeneous anisotropic media, we hope that it might be substantiated in the future when the relevant results become available.

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