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Low-frequency symmetric waves in fluid-filled boreholes and pipes with radial layering

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ABSTRACT

Many tasks in geophysics and acoustics require estimation of mode velocities in cylindrically layered media. For example, acoustic logging or monitoring in open and cased boreholes need to account for radial inhomogeneity caused by layers inside the borehole (sand screen, gravel pack, casing) as well as layers outside (cement, altered and unaltered formation layers). For these purposes it is convenient to study a general model of cylindrically layered media with inner fluid layer and free surface on the outside. Unbounded surrounding media can be described as a limiting case of this general model when thickness of the outer layer is infinite. At low frequencies such composite media support two symmetric modes called Stoneley (tube) and plate (extensional) wave. Simple expressions are obtained for these two mode velocities valid at zero frequency. They are written in a general form using elements of a propagator matrix describing axisymmetric waves in the entire layered composite. This allows one to apply the same formalism and compute velocities for n-layered composites as well as anisotropic pipes. It is demonstrated that the model of periodical cylindrical layers is equivalent to a homogeneous radially transversely isotropic media when the number of periods increases to infinity, whereas their thickness goes to zero. Numerical examples confirm good validity of obtained expressions and suggest that even small number of periods may already be well described by equivalent homogeneous anisotropic media.

INTRODUCTION

Propagation of symmetric modes in cylindrically layered materials is of significant interest to many practical applications in acoustics and borehole geophysics (White 1983; Bakulin *et al.* 2008). The most established method for computing speeds of acoustic modes is the so-called root-finding technique (Ewing, Jardetzky and Press 1957). However, this method as well as other techniques become very tedious for analytical analysis when the number of layers increases. While some of these techniques can be efficiently programmed on a computer, this provides little physical insight into the problem at hand. In this paper we focus on deriving wave speeds of symmetric modes in a low-frequency limit in cylindrically layered elastic media with inner fluid layer and vacuum outside. Each solid layer is described as linear elastic material with constants that may vary only as a function of radial coordinate. In case of a layered media, we assume welded contact between solid layers. Such structures support two fundamental wave modes that exist from a zero frequency (Del Grosso 1971; Lafleur and Shields 1995; Bakulin *et al.* 2008): Stoneley (tube) wave and plate (extensional) wave. We obtain formulae for their velocities at zero frequency that require knowledge of six elements of a cylindrical propagator matrix for axisymmetric

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waves. These formulae are valid for a broad class of propagator matrices. We provide propagator matrices in the low-frequency limit for homogeneous isotropic and radially transversely isotropic materials. We also present a formalism that allows obtaining a low-frequency propagator matrices for a composite media with an arbitrary number of elastic layers with welded contact. In section 'Mode veolcities in fluid-filled elastic pipe' we derive a dispersion equation for symmetric modes in a general case of a radially inhomogeneous material that may also be radially anisotropic. With respect to squared slowness, it is a quadratic equation with coefficients expressed through some elements of the propagator matrix as well as fluid parameters. We further present approximate solutions for special cases of thin and thick pipes. In section 'Velocities for isotropic, layered and anisotropic pipes' we apply the general dispersion equation to the cases of homogeneous, layered and anisotropic media. We also present numerical computations to illustrate good applicability of derived formulae for velocities. In particular, we apply them to a case of periodically layered media with small and large thicknesses. We observe that periodically layered media can be replaced by an equivalent homogeneous media with anisotropic behaviour.

In Appendix A we construct a low-frequency propagator matrix for axisymmetric waves in layered media from corresponding matrices of the layers. We then consider periodical systems. We show that in the limiting case of infinitely large numbers of periods, such a propagator matrix acquires a new structure that is identical to that of a radially transversely isotropic homogeneous material. In Appendix B we outline one possible way to derive apropagator matrix for axisymmetric waves in the limit of low frequencies. This method follows a recipe from Molotkov (Petrashen, Molotkov and Krauklis 1985). Finally, Appendix C outlines a possible extension when a pipe contains both fluid and solid layers.

MODE VELOCITIES IN FLUID-FILLED ELASTIC PIPES

In this section we consider an elastic pipe with fluid inside and vacuum outside and obtain expressions for mode velocities in the zero-frequency limit. To derive a dispersion equation we tie solutions in the fluid and in the elastic media using continuity of radial displacement and normal stresses at the boundary between the fluid and the solid. Another simpler but approximate method for obtaining one of the velocities is considered as a generalization of White's approach (White 1983).

General matrix solution of elastic equations

Consider a linear elastic material with cylindrical symmetry that can be radially anisotropic, and inhomogeneous only in a radial direction. As shown in Appendix B, after transferring time t and coordinate z to frequency ω and axial wavenumber ξ, we obtain a system of two differential equations of the second order for amplitudes $u_r(r)$ and $u_z(r)$. This equation has four independent solutions. Let us define $\sigma_{rr}(r)$ and $\sigma_{rz}(r)$ as radial and tangential stresses, respectively. While using stresses and displacements is natural in rectangular coordinates, in the cylindrical case it is more convenient to use combinations ru_r and $r\sigma_{rz}$ and define the following vector

$$W^t = [u_z \quad \sigma_{rr} \quad ru_r \quad r\sigma_{rz}]. \tag{1}$$

Appendix B shows that solutions at two different radii are related as

$$W(r) = G(r, r_0)\, W(r_0), \tag{2}$$

where G is the propagator matrix of the layer that 'propagates' vector W from one radius to another. This last equation describes an elastic pipe with arbitrary boundary conditions on its sides.

Fluid-filled pipes in vacuum

To consider fluid-filled pipes in vacuum we need to apply appropriate boundary conditions. Let us denote the pipe's inner radius as r_0 and the outer radius as r. Boundary conditions then become

$$\begin{aligned} &\sigma_{rr}(r) = \sigma_{rz}(r) = 0, \\ &\sigma_{rr}(r_0) = -p, \qquad \sigma_{rz}(r_0) = 0, \qquad u_r(r_0) = u, \end{aligned} \tag{3}$$

where p is fluid pressure and u is radial displacement of the fluid at the boundary with the pipe. Thus equation (2) is replaced by

$$\begin{bmatrix} u_z(r) \\ 0 \\ ru_r(r) \\ 0 \end{bmatrix} = G(r, r_0) \begin{bmatrix} u_z(r_0) \\ -p \\ r_0 u \\ 0 \end{bmatrix}. \tag{4}$$

To obtain a dispersion equation for symmetric modes we only need two equations from this matrix system:

$$\begin{cases} 0 = g_{21}\, u_z(r_0) - g_{22}\, p + g_{23} r_0 u, \\ 0 = g_{41}\, u_z(r_0) - g_{42}\, p + g_{43} r_0 u. \end{cases} \tag{5}$$

Elimination of $u_z(r_0)$ yields the desired dispersion equation

$$-g_{41}g_{22} + g_{41}g_{23}\frac{u}{p}r_0 = -g_{21}g_{42} + g_{21}g_{43}\frac{u}{p}r_0. \tag{6}$$

On the one hand, this equation contains elements of G which represent the solution of dynamic equations for solid media. On the other hand, it contains the ratio $\frac{u}{p}$ determined on the boundary with the fluid, and thus represents the solution of the fluid dynamic equation. Therefore equation (6) relates solutions for fluid and solid media and provides a dispersion equation (g_{ij} and u/p depend on the phase velocity). This dispersion equation is valid for any frequency and can be applied to any material for which propagator matrix G can be constructed. For example, it can be used even for a multilayered elastic material with nonviscous fluid layers inside it (see Appendix C). In the remainder of this article we use this general equation to study low-frequency velocities of symmetric modes in composite pipes of various nature.

Dispersion equation at zero frequency

To derive the dispersion equation for axisymmetric modes at zero frequency for a fluid-filled pipe in vacuum, we need to specify how coefficients in equation (6) depend on phase velocity and the wavenumber ξ. We assume that boundary conditions (3) are fulfilled. Let us handle the elastic pipe first and the internal fluid column later.

Let us consider a wavelength that is large compared to the pipe's outer radius $\xi r \ll 1$. Although elements g_{ij} depend on ω and ξ (which come from ∂_t and ∂_z), it is convenient to use phase velocity $c = \omega/\xi < \infty$ instead of ω and assume that c is finite.

Proposition 1. In the low-wavenumber limit, elements of matrix G required for the dispersion equation have the following structure

$$g_{21} = q_{21}\xi + O(\xi^3), \qquad g_{2l} = q_{2l} + O(\xi^2),$$

$$g_{41} = \left[-\rho c^2 \frac{yr_0^2}{2} + q_{41}\right]\xi^2 + O(\xi^4), \qquad g_{4l} = q_{4l}\xi + O(\xi^3),$$

where $l = 2, 3$; $y = \frac{r^2}{r_0^2} - 1$, and q_{ij} are independent of c.

Proof. While complete proof can only be given when propagator matrix G is fully defined, part of the result can be easily explained without complete definition. The form of g_{22} follows from the obvious property of the propagator matrix $G(r, r) = I$, where I is a unit matrix. The expression for g_{41} can be obtained directly from the equation of motion

$$-\rho\omega^2 u_z(r) = \frac{1}{r}\partial_r(r\sigma_{rz}(r)) - \xi\sigma_{zz}(r). \tag{7}$$

Indeed, parameter ξr is small and the wavelength is much greater then the pipe thickness. Therefore $u_z(r_1) \simeq u_z(r_0)$ for any internal radius $r_0 \le r_1 \le r$. Utilizing this approximation and selecting boundary conditions $u_r(r_0) = \sigma_{rr}(r_0) = r_0\,\sigma_{rz}(r_0) = 0$, we obtain after integration

$$r\sigma_{rz}(r) = -\rho c^2\xi^2\frac{yr_0^2}{2}u_z(r_0) + \xi\int_{r_0}^{r} dr_1 r_1\sigma_{zz}(r_1).$$

For linearly elastic materials we can further assume $\sigma_{zz} \sim \xi u_z(r_0)$. On the other hand, under the same boundary conditions, equation (2) gives $r\sigma_{rz}(r) = g_{41}u_z(r_0)$. Comparing both expressions reveals the required structure of g_{41}.

In the homogeneous isotropic case, the structure of other elements easily follows from equations (B13) and (B19). □

Now let us clarify the dependence of fluid-related quantity u/p on phase velocity c. Such a relation can be derived from the solution of the wave equation for fluid media in the zero-wavenumber limit (see equation (C3) in Appendix C). At the cylindrical boundary $r = r_0$ inside the fluid layer we have

$$\frac{u}{p} = \left(\frac{v_f^2}{c^2} - 1\right)\frac{r_0}{2K} = (X-1)\frac{r_0}{2K}, \tag{8}$$

where v_f and K are the longitudinal velocity and bulk modulus of the fluid respectively; $X = v_f^2/c^2$ is normalized squared slowness.

It is convenient to write the dispersion equation with respect to X rather than to c. Let us also introduce elastic parameter μ and density parameter ρ. In case of isotropic homogeneous media they have the physical meaning of shear modulus and density of the pipe material, respectively. For more complex pipes ρ becomes volume-averaged density. Then it is natural to work with nondimensional parameters β and γ defined as

$$\beta = \frac{\rho v_f^2}{\mu}, \qquad \gamma = \frac{K}{\mu}. \tag{9}$$

(For homogeneous isotropic media they coincide with β and γ used by Lafleur and Shields (1995)). Then using proposition 1 and equation (8), we can rewrite the dispersion equation (6) as

$$\left(-\frac{\mu\beta}{X}\frac{yr_0^2}{2} + q_{41}\right)\left[-q_{22}\,\mu\gamma + q_{23}\frac{r_0^2}{2}(X-1)\right] = q_{21}\left[-q_{42}\,\mu\gamma + q_{43}\frac{r_0^2}{2}(X-1)\right]. \tag{10}$$

Equation (10) can be rewritten in the form

$$EX^2 + FX + G = 0, \tag{11}$$

with coefficients

$$E = q_{41}q_{23} - q_{43}q_{21}, \tag{12a}$$

$$F = \mu\gamma[q_{42}q_{21} - q_{41}q_{22}]\frac{2}{r_0^2} - E - \mu\beta q_{23}\frac{yr_0^2}{2}, \tag{12b}$$

$$G = \mu^2\beta\gamma q_{22}y + \mu\beta q_{23}\frac{yr_0^2}{2}, \tag{12c}$$

where as before $y = \frac{r^2}{r_0^2} - 1$.

Quadratic form of equation (11) suggests that the system supports two axisymmetric waves. These two waves can be considered as generalized tube and plate waves (Bakulin *et al.* 2008), since one is formed mainly by the fluid column, and the other - by the pipe. However, this division becomes rather ambiguous when two roots are close to each other.

Approximate solutions of the dispersion equation

After inspection of equation (12) we can see that coefficients E, F and G have common parts. Thus it is natural to collect these common terms and introduce quantities

$$g_1 = \mu^2\beta\gamma q_{22}y, \qquad g_2 = \mu\beta q_{23}\frac{yr_0^2}{2}, \tag{13a}$$

$$f = \mu[q_{41}q_{22} - q_{42}q_{21}]\frac{2}{r_0^2}. \tag{13b}$$

Then coefficients F and G can be rewritten as

$$F = -\gamma f - E - g_2 \qquad \text{and} \qquad G = g_1 + g_2. \tag{14}$$

If we introduce

$$B = -F - 2g_2 = E + \gamma f - g_2, \tag{15}$$

then the exact solution of the dispersion equation can be written as

$$X = \frac{B + 2g_2 \pm \sqrt{B^2 + 4(\gamma f g_2 - Eg_1)}}{2E}. \tag{16}$$

Under the condition

$$B^2 \gg |\gamma f g_2 - Eg_1|, \tag{17}$$

these solutions can be simplified. As shown in section 'Homogeneous isotropic pipe', for the homogeneous and isotropic pipe this condition is satisfied for thick ($y \gg 1$) and thin ($y \ll 1$) pipes. For convenience, let us consider the case when also

$$B = E + \gamma f - g_2 > 0. \tag{18}$$

Expanding the square root and applying equation (17), we obtain the following simple approximations for squared normalized slownesses X and phase velocities $c = v_f/\sqrt{X}$:

$$X_+ = 1 + \gamma\frac{f}{E}, \qquad C_+ = \frac{v_f}{\sqrt{1 + \gamma\frac{f}{E}}}, \tag{19a}$$

$$X_- = \frac{EG - g_2^2}{BE}, \qquad C_-^2 = \frac{\mu}{\rho}\frac{\beta BE}{EG - g_2^2}. \tag{19b}$$

Quasistatic approximation (White's approach)

White (1983) suggested an alternative method to obtain the tube-wave velocity C_t in the quasistatic limit. It follows from equation (6) and proposition 1 that

$$\frac{p}{u} = \frac{2M}{r_0}, \quad \text{where} \quad 2M = \frac{q_{41}q_{23} - q_{43}q_{21}}{q_{41}q_{22} - q_{42}q_{21}}r_0^2. \tag{20}$$

On the other hand, we can express ratio $\frac{\hat{u}(z,t)}{\hat{p}(z,t)}$ at the same boundary $r = r_0$ but inside the fluid. Equating $\frac{u}{p} = \frac{\hat{u}}{\hat{p}}$ gives the dispersion equation. The equation of motion for the fluid medium is

$$\frac{\partial}{\partial z}\hat{p}(z,t) = -\rho_f\frac{\partial^2}{\partial t^2}u_z(z,t), \tag{21}$$

where $\hat{p}$, ρ_f and u_z are pressure, density and axial displacement in the fluid at r_0, respectively. Pressure can be found from the continuity equation. In the quasistatic or low-frequency limit ($\xi r \ll 1$) we can write $\frac{u}{r_0} \simeq \partial_r u_r(r_0)$ and thus, using equation (20), obtain

$$\hat{p} \simeq -K\left[\frac{\partial u_z}{\partial z} + \frac{2\hat{u}}{r_0}\right] = -K\left[\frac{\partial u_z}{\partial z} + \frac{\hat{p}}{M}\right], \tag{22}$$

where K is the bulk modulus of the fluid and $\hat{u} = u_r(r_0, z, t)$. White (1983) wrote this equation as $p \simeq -K\frac{\Delta V}{V}$ in terms of relative volume change for a small fluid cylinder at hand. Differentiating equation (22) along the axial coordinate, we can express $\partial_z\hat{p}$ as

$$\left[\frac{1}{K} + \frac{1}{M}\right]\frac{\partial\hat{p}}{\partial z} = -\frac{\partial^2 u_z}{\partial z^2}. \tag{23}$$

Substituting this expression into equation (21), we obtain

$$\frac{\partial^2 u_z}{\partial z^2} = \rho_f\left[\frac{1}{M} + \frac{1}{K}\right]\frac{\partial^2 u_z}{\partial t^2}, \tag{24}$$

which is a one-dimensional wave equation with phase velocity (White 1983)

$$C_t = \frac{v_f}{\sqrt{1 + \frac{K}{M}}}. \tag{25}$$

This approximate expression for the phase velocity coincides with equation (19a). This can be verified by noticing that

$$\frac{f}{E} = \frac{\mu}{M} \quad \text{and} \quad \gamma = \frac{K}{\mu}, \tag{26}$$

which follows from equations (12a), (13b), (20) and (9).

White's quasistatic approach can be justified using exact dispersion equation equation (10) by adding an explicit assumption of smallness of velocity C_t as compared to the second root or plate-wave velocity C_{pl}. As will be shown in equation (42), for a thin pipe ($h \ll r_0$) velocity C_{pl} can be found as a root of the simple equation $g_{41} = 0$. When finding an expression for a second smaller root $c = C_t$ under the assumption $C_t^2 \ll C_{pl}^2$, we can neglect the first term in g_{41} and write

$$g_{41} \simeq \left[-\rho c^2 \frac{yr_0^2}{2} + q_{41} \right] \xi^2 \simeq q_{41} \xi^2.$$

This corresponds to neglecting the first term $\frac{\mu\beta}{X}\frac{yr_0^2}{2}$ in parentheses of equation (10). After such simplification, the quadratic with respect to X equation (10) turns into linear equation equation (19a) for the tube-wave slowness identical to White's equation (25). One has to be aware of additional assumptions behind White's low-frequency equation. For example, White's equation can be particularly misleading when C_t and C_{pl} become close to each other (Bakulin *et al.* 2008).

White described only an isotropic and homogeneous pipe whereas Norris (1990) extended this quasistatic approach to treat additional borehole environments with acoustic logging tools and casing. In this paper we generalize the quasistatic approach by explaining how quantity M (equation (20)) can be calculated for the case of radially anisotropic inhomogeneous pipe, if its propagator matrix G is known.

VELOCITIES FOR LAYERED ISOTROPIC AND ANISOTROPIC PIPES

Let us apply the presented formalism to cases of homogeneous isotropic, layered and radially anisotropic elastic pipes. As in all the cases considered before, the pipe is filled with fluid on the inside and has vacuum outside. We also give simplified approximations for thin and thick pipes.

Homogeneous isotropic pipes

Let us consider an isotropic pipe with density ρ and Lamé parameters λ and μ. The derivation of the propagator matrix for this case is described in Appendix B. Here we only present the final result.

Proposition 2. For homogeneous isotropic elastic media the low-wavenumber limit of the propagator matrix $G(r, r_0)$ is given by

$$\begin{bmatrix} 1 & \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\left(\frac{y}{2} - \ln\frac{r}{r_0}\right) r_0^2 \xi & \frac{\lambda+\mu}{\lambda+2\mu}\frac{y}{2}\xi + \frac{\mu}{\lambda+2\mu}\xi \ln\frac{r}{r_0} & \frac{1}{\mu}\ln\frac{r}{r_0} \\ \frac{\lambda\mu}{\lambda+2\mu} x\xi & 1 - \frac{\mu}{\lambda+2\mu} x & \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\frac{2x}{r_0^2} & -\frac{\lambda+\mu}{\lambda+2\mu}\frac{x}{2}\xi - \frac{\mu}{\lambda+2\mu}\xi \ln\frac{r}{r_0} \\ -\frac{\lambda}{\lambda+2\mu}\frac{yr_0^2}{2}\xi & \frac{1}{\lambda+2\mu}\frac{yr_0^2}{2} & 1 + \frac{\mu}{\lambda+2\mu} y & \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\left(\frac{x}{2} - \ln\frac{r}{r_0}\right) r^2 \xi \\ \left(-\rho c^2 + \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\right)\frac{yr_0^2}{2}\xi^2 & \frac{\lambda}{\lambda+2\mu}\frac{yr_0^2}{2}\xi & \frac{\lambda\mu}{\lambda+2\mu} y\xi & 1 \end{bmatrix}, \tag{27}$$

where $\xi = \frac{\omega}{c}$ is a wavenumber, c ($c < \infty$) is a phase velocity, $x = 1 - (\frac{r_0}{r})^2$ and $y = (\frac{r}{r_0})^2 - 1$ are two alternative forms of a geometric parameter.

To calculate coefficients of the dispersion equation (11), we should compare elements g_{ij} of matrix (27) with their representation in proposition 1 and find q_{ij}. In this particular case it is convenient to multiply coefficients of equation (11) by a factor $2(1-\sigma)/(x\mu^2)$ and use expressions involving Poisson's ratio σ

$$\sigma = \frac{\lambda}{2(\lambda+\mu)}, \quad 1+\sigma = \frac{3\lambda+2\mu}{2(\lambda+\mu)}, \quad 1-\sigma = \frac{\lambda+2\mu}{2(\lambda+\mu)}.$$

After introduction of modified coefficients and some simplification, phase velocities $c = v_f/\sqrt{X}$ can be found from the equation

$$E'X^2 + F'X + G' = 0, \tag{28}$$

with coefficients

$$E' = 2(1-\sigma)\frac{1}{x}E = 2y(1+\sigma), \tag{29a}$$

$$F' = -4\gamma - 2y(1+\gamma)(1+\sigma) - \beta y, \tag{29b}$$

$$G' = 2\beta\gamma(1-\sigma) + y\beta(1+\gamma). \tag{29c}$$

This equation was obtained by Lafleur and Shields (1995) (although they used parameter $\alpha = \frac{\rho v_f^2}{\lambda+2\mu}$ instead of σ).

Approximations for thin and thick isotropic pipe

We utilize equations (19a) and (19b), but as before introduce modified coefficients B', E', G', g'_2 and f' multiplied by

$2(1-\sigma)/(x\mu^2)$ that are more convenient for isotropic pipe

$$f' = 4 + 2y(1+\sigma) = 4 + E',$$
$$g_2' = y\beta,$$
$$B' = E' + \gamma f' - g_2' = (1+\gamma)E' + 4\gamma - y\beta\ .$$

Then the roots of equation (16) are

$$X = \frac{B' + 2y\beta \pm \sqrt{B'^2 + 4y\beta\gamma\sigma^2}}{2E'}. \tag{30}$$

With all coefficients now defined for isotropic pipe, we can clarify that condition (17) is met if the pipe is thin ($y \ll 1$) or thick ($y \gg 1$). Indeed for a thin pipe we have $B'^2 \simeq 16\gamma^2 \gg y$. Likewise for a thick pipe $B'^2 \sim y^2 \gg y$. Therefore tube-wave velocity approximation (19a) can be written for isotropic pipe as

$$C_t = \frac{v_f}{\sqrt{1 + \frac{K}{\mu}\left[1 + \frac{2}{y(1+\sigma)}\right]}}. \tag{31}$$

When $y \to \infty$ we obtain the velocity for a well-known case of fluid-filled borehole surrounded by an infinite elastic medium (White 1983)

$$X_t^{(\infty)} = 1 + \frac{K}{\mu}, \quad C_t^{(\infty)} = \frac{v_f}{\sqrt{1+\frac{K}{\mu}}}. \tag{32}$$

The expression for the plate velocity is rather complex and therefore we only write it for the cases of thin and thick pipes. For thin pipe we have

$$y \to 0, \quad g_2' \to 0, \quad G' \approx 2\beta\gamma(1-\sigma), \quad B' \approx 4\gamma,$$

which gives

$$X_{pl}^{(0)} = \frac{G'}{B'} = \frac{\beta(1-\sigma)}{2} \quad \text{and} \quad C_{pl}^{(0)} = v_s\sqrt{\frac{2}{1-\sigma}}, \tag{33}$$

where v_s is the shear-wave velocity of the pipe material. In the thin-pipe limit, the tube-wave velocity goes to zero, whereas the plate-wave velocity remains finite.

For the thick pipe we have $y \to \infty$ and

$$g_2' = y\beta, \quad G' \approx y\beta(1+\gamma), \quad B' \approx E'(1+\gamma) - y\beta.$$

Substituting these expressions into (19b), we obtain

$$X_{pl}^{(\infty)} = \frac{\beta}{2(1+\sigma)} \quad \text{and} \quad C_{pl}^{(\infty)} = \sqrt{\frac{N}{\rho}}, \tag{34}$$

where N and ρ are Young's modulus and density of the pipe material. Note that the pipe can be considered as thick when its outer radius is finite (and even small) and the inner radius vanishes ($r_0 \ll r$). In a limiting case of $r_0 = 0$ we obtain extensional mode velocity for a solid rod.

In all these approximations we used the condition $B' > 0$ to simplify the square root $\sqrt{B'^2} = |B'|$. This condition is satisfied in many practical cases and it turns out that the even less restrictive condition

$$\sqrt{\frac{N}{\rho}} > \frac{v_f}{\sqrt{1+\frac{K}{\mu}}} \tag{35}$$

is often sufficient. It can be rewritten as $X_t^{(\infty)} > X_{pl}^{(\infty)}$ and tells us that plate-wave velocity in the solid rod should exceed tube-wave velocity in the borehole surrounded by an infinite medium of the same material. This condition is sufficient to make B' positive, since

$$B' = 4\gamma + 2y(1+\sigma)\left[X_t^{(\infty)} - X_{pl}^{(\infty)}\right]. \tag{36}$$

Moreover, when condition (35) is fulfilled, approximation (31) can be used for any y (not only for small or large y), although it provides less accuracy.

Finally, we emphasize that while White (1983) originally postulated only quasistatic conditions (large wavelength), we can see that additional assumptions about material or geometric parameters are implicit in his approach and in the final expression (31).

Radially layered pipe

Let us now apply the developed formalism to a pipe consisting of n homogeneous isotropic elastic layers. In order to utilize the dispersion equation, we need to establish a matrix propagator for the pipe as a whole.

Let us introduce additional notations. The layer with number i has inner radius r_{i-1}, outer radius r_i, material parameters λ_i, μ_i, ρ_i and geometric parameters

$$x_i = 1 - \frac{r_{i-1}^2}{r_i^2}, \qquad y_i = \frac{r_i^2}{r_{i-1}^2} - 1. \tag{37}$$

Propagator matrix G for the layered pipe is

$$G(r_n, r_0) = G_n(r_n, r_{n-1}) \dots G_2(r_2, r_1)G_1(r_1, r_0), \tag{38}$$

where propagators $G_j (j = 1, \dots, n)$ are expressed as equation (27) for each homogeneous layer. For convenience, in the following text we denote $r = r_n$ for any number of layers. We can compute matrix products and represent elements of G as series in the wavenumber ξ. Since we are only interested in a limit, we retain only the first terms of the series representing each element. After higher-order terms are neglected, each element of the propagator matrix can be decomposed into two factors. One is geometric (depends only on radii) and the other is defined by material properties. This can be done for propagators of single homogeneous layers as well as for propagators describing the layered pipe. In the layered case, the geometric factor is the same and contains only the inner and outer radii of the pipe, whereas the property-factor of each element represents an effective material constant that depends on the properties and geometry of individual layers.

It turns out that the propagator matrix for a multilayered pipe ($n \geq 1$) preserves the structure seen for the homogeneous isotropic matrix (27), albeit with a larger number of material parameters. Multilayered propagator can be expressed as

$$\begin{bmatrix} 1 & q_{12}\xi & q_{13}\xi & l_{14}\ln\frac{r}{r_0} \\ s_{21}x\xi & 1+(s_{22}-1)x & s_{23}\frac{2x}{r_0^2} & q_{24}\xi \\ s_{31}\frac{yr_0^2}{2}\xi & s_{32}\frac{yr_0^2}{2} & 1+s_{33}y & q_{34}\xi \\ (-\rho c^2+s_{41})\frac{yr_0^2}{2}\xi^2 & s_{42}\frac{yr_0^2}{2}\xi & s_{43}y\xi & 1 \end{bmatrix}, \tag{39a}$$

while remaining elements are further broken down and their terms separately decomposed as

$$\begin{aligned} q_{12} &= \left[s_{12}y + 2l_{12}\ln\frac{r}{r_0}\right]\frac{r_0^2}{4},\ q_{13} = s_{13}\frac{y}{2} + l_{13}\ln\frac{r}{r_0}, \\ q_{34} &= \left[s_{34}x + 2l_{34}\ln\frac{r}{r_0}\right]\frac{r^2}{4},\ q_{24} = s_{24}\frac{x}{2} + l_{24}\ln\frac{r}{r_0}. \end{aligned} \tag{39b}$$

We use two sets of property factors (s_{ij} and l_{ij}) to emphasize that l_{ij} are multiplied by logarithms. The factor ρ is volume averaged density. Note that the stated powers of ξ in the first terms reflect the general structure of linear elasticity equations and thus remain valid for radially anisotropic and inhomogeneous cases. In the Appendix A we provide explicit equations for property factors s_{ij} and l_{ij} in several special cases.

Once the propagator matrix is established, we can obtain formulae for velocities by substituting elements of matrix (39a) into (20). Recalling that $q_{22} = 1 + (s_{22} - 1)x$ and $q_{41} = s_{41}\frac{yr_0^2}{2}$, we obtain

$$M = \frac{[s_{41}s_{23} - s_{43}s_{21}]y}{s_{41} + [s_{41}s_{22} - s_{42}s_{21}]y}. \tag{40}$$

If we define 'effective' shear modulus μ and 'effective' Poisson ratio σ as

$$\mu = \frac{s_{41}s_{23} - s_{43}s_{21}}{s_{41}s_{22} - s_{42}s_{21}}, \quad 1+\sigma = \frac{2}{s_{41}}[s_{41}s_{22} - s_{42}s_{21}],$$

then we can express the constant M as $M = \frac{\mu(1+\sigma)y}{2+(1+\sigma)y}$. This gives a familiar approximation for tube-wave velocity

$$C_t = \frac{v_f}{\sqrt{1+\frac{K}{M}}} = \frac{v_f}{\sqrt{1+\frac{K}{\mu}\left[1+\frac{2}{y(1+\sigma)}\right]}}, \tag{41}$$

that has form identical to that of isotropic homogeneous pipe (see equation (31)). Note that μ and σ have a meaning of effective parameters and depend on the properties and geometry of all layers. The geometric factor appearing in the final equation characterizes the entire pipe $y = \frac{r^2}{r_0^2} - 1$.

In the limit of thin pipe ($y \ll 1$), the plate wave velocity is given by a very simple expression

$$C_{pl}^{(0)} = \sqrt{\frac{s_{41}}{\rho}}, \tag{42}$$

which is derived from equation (19b). This expression shows that for a thin pipe, the plate-wave velocity is controlled by the property factor of a single element of matrix G and can be found from equation $g_{41} = 0$ as seen from equation (39a).

Pipe characterized by radial transverse isotropy

Let us now examine the propagator matrix for a radially anisotropic pipe. While in principle, materials as complex as radially orthotropic can be handled, here we focus on a simpler case of radial transverse isotropy because it describes finely layered periodical pipe made of isotropic layers. This is proved in Appendix A and illustrated numerically in section 'Numerical examples'. Here we only describe the propagator matrix for a homogeneous cylindrical layer of such material. As pointed out by Love (1944) radial transverse isotropy has a property that in every point inside the pipe symmetry axis tracks the radial direction. Therefore Hooke's law in cylindrical coordinates can be written as

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\phi\phi} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{13} & c_{13} \\ c_{13} & c_{33} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\phi\phi} \\ \epsilon_{zz} \end{bmatrix}, \qquad \sigma_{rz} = c_{44}\epsilon_{rz},$$
$$\sigma_{r\phi} = c_{44}\epsilon_{r\phi}, \qquad \sigma_{\phi z} = \frac{c_{33} - c_{23}}{2}\epsilon_{\phi z}. \tag{43}$$

Let us introduce anisotropy parameter $\nu^2 = \frac{c_{33}}{c_{11}}$, and define additional geometry-property factors as

$$y_+ = \left(\frac{r}{r_0}\right)^{1+\nu} - 1, \qquad y_- = \left(\frac{r}{r_0}\right)^{1-\nu} - 1. \tag{44}$$

As the propagator matrix G for such a medium is rather complex, let us first introduce auxiliary matrix $T(\nu)$ with elements $t_{ij}(r, r_0; \nu)$. The second column of T is

$$\begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \\ t_{42} \end{bmatrix} = \begin{bmatrix} \frac{\xi r_0^2\left(\left[c_{44}+\frac{(c_{23}+\nu c_{13})}{1+\nu}\right]y_+ - (c_{23}+\nu c_{13})\ln\frac{r}{r_0}\right)}{2\nu(1+\nu)c_{11}c_{44}} \\ \left(1+\frac{c_{13}}{\nu c_{11}}\right)\frac{(y_+ - y)r_0^2}{2r^2} \\ \frac{r_0^2 y_+}{2\nu c_{11}} \\ \frac{(c_{23}+\nu c_{13})\xi r_0^2 y_+}{2\nu(1+\nu)c_{11}} \end{bmatrix}. \tag{45a}$$

The remaining elements are

$$t_{21}(r,r_0) = -t_{43}(r_0,r), \qquad t_{24}(r,r_0) = t_{13}(r_0,r),$$
$$t_{31}(r,r_0) = t_{42}(r_0,r), \qquad t_{34}(r,r_0) = -t_{12}(r_0,r), \tag{45b}$$

$$t_{j3}(\nu) = t_{j2}(\nu)\frac{\nu c_{11} - c_{13}}{r_0^2}, \qquad t_{14} = \frac{1}{c_{44}}\ln\frac{r}{r_0},$$
$$t_{11} = t_{44} = 0,$$
$$t_{41} = \frac{\xi^2 y r_0^2}{4}\left(-\rho c^2 + c_{33} - \frac{c_{13}^2}{c_{11}}\right) + \xi^2 r_0^2\left(\frac{y}{2} - \frac{y_+}{1+\nu}\right)\frac{\left(c_{23}^2 - \nu^2 c_{13}^2\right)}{2\nu(1-\nu)c_{11}}. \tag{45c}$$

Then the propagator matrix is expressed as

$$G = I + T(\nu) + T(-\nu). \tag{45d}$$

For example, element g_{32} is given as $g_{32} = \frac{r_0^2(y_+ - y_-)}{2\nu c_{11}}$. Note that the symmetry relationships (45b) are the same for matrices T and G.

It should be noted that the structure of the anisotropic propagator is distinct from the isotropic (equation (27)) or layered case (equation (39)) due to the presence of the factors y_+ and y_- that couple the geometry and material parameters. We later show that this new structure can be obtained as a limiting case of a layered isotropic propagator when $n \to \infty$. The isotropic propagator matrix (27) can be obtained as a limiting case of the anisotropic one if we take into account that

$$c_{11} = c_{33}, \qquad c_{13} = c_{23}, \qquad c_{44} = \tfrac{1}{2}(c_{33} - c_{23}),$$
$$\nu = 1, \qquad y_+ = y,\ y_- = 0, \qquad \tfrac{y_-}{1-\nu} \xrightarrow{\nu \to 1} \ln\tfrac{r}{r_0}.$$

Once the anisotropic propagator (45) is established, the same formalism can be applied to obtain dispersion equation and mode velocities for radially transversely isotropic pipe. For example, in order to use equations (41) and (42), one needs to equate the elements of anisotropic propagator (equation (45)) with the template (39a). Expressing six stiffnesses s_{ij} from equation (40) and compliance s_{41} required for computation of tube- and plate-wave velocities respectively, one can use the same approximate equations (41) and (42) as before. Note that factors y_+ and y_- will be absorbed by s_{ij}. Likewise, to use exact dispersion equation (10) one needs to equate elements g_{ij} listed in proposition 1 with corresponding elements of anisotropic propagator (45) and define required q_{ij}. We omit the detailed expressions for brevity and only demonstrate them in the numerical examples.

Table 1 Parameters of two-component periodic models used for computations

Model	A	B	C	
r_0/r	9/10	5/10	5/10	
θ	1/2*	2/3†	2/3†	
n	5	10	5	
Material	PVC	Steel 1	Steel 2	Water
ρ (kg/m^3)	1400	7870	7870	1000
$V_p (m/s)$	2020	5600	5600	1500
$V_s (m/s)$	945	3190	399	0

* Linear (by thickness) concentration of PVC.
† Volume concentration of PVC.

Table 2 Comparison of velocities at zero frequency computed using different methods explained in the text

	Matrix multiplication	Anisotropic	Average	Spectral method‡
Model A				
$V_t (m/s)$	1243	1247	1253	1241
$V_{pl} (m/s)$	4752	4749	4753	4747
Model B				
$V_t (m/s)$	1336	1359	1410	1336
$V_{pl} (m/s)$	4427	4428	4425	4428
Model C				
$V_t (m/s)$	764	764	764	761
$V_{pl} (m/s)$	1046	1046	1046	1045

‡ Computed at finite frequency where results become stable.

Numerical examples

Let us illustrate the behaviour of the velocities in fluid-filled layered pipes and their dependence on various parameters. In particular, we consider periodically layered pipes with periods of equal thickness. Each period consists of two sublayers. Total thickness of the pipe is denoted as h and its inner and outer radii are r_0 and r. We have water inside and vacuum outside the pipe.

We consider three models: A, B and C. In models B and C the first inner sublayer is polyvinyl chloride (PVC) with constant volume concentration θ across all periods. Volume concentration is defined as the ratio of the cross-sectional area of the PVC layer to the entire cross-section area of the period. In model A concentration θ is equal to the ratio of the thicknesses, however due to small overall thickness of the pipe

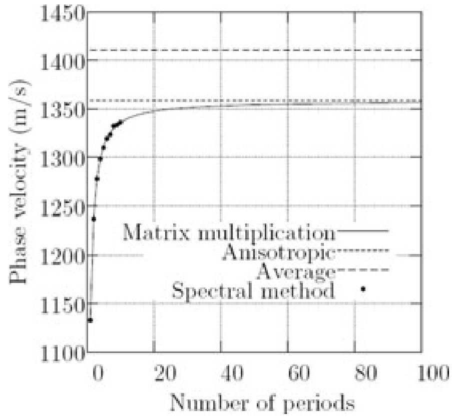

Figure 1 Tube-wave velocity in Model B as a function of the number of periods of constant thickness. Shown are velocities obtained by different methods explained in the text.

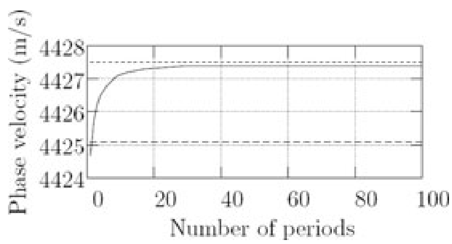

Figure 2 Plate-wave velocity in Model B as a function of the number of periods of constant thickness. Same notations as Fig. 1.

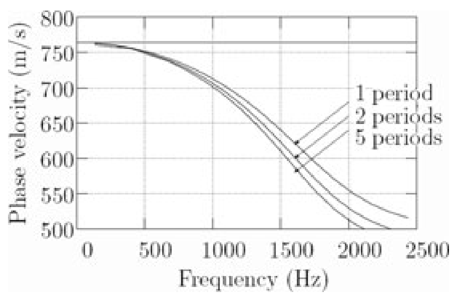

Figure 3 Tube-wave velocity in Model C. Since correction terms are zero (equation (A22)), all methods produce the same velocities, shown by horizontal solid line irrespective of number of layers. Thin lines denote actual dispersion curves obtained with the spectral method (Karpfinger *et al.* 2007), which collapse to the same value of velocity at zero frequency.

it also implies approximately equal volume concentrations. The second sublayer is steel in models A and B, but in model C it is steel with a modified shear modulus equal to that of PVC (Table 1).

Results are shown in Table 2 and on Figs 1–3 and display velocities obtained by four different methods. 'Average' denotes wavespeeds obtained from a propagator that represents simple volume averaging of the propagators for the constituent layers (equation (A29)). 'Matrix multiplication' denotes velocities obtained using exact propagator multiplication as in equation (38). To verify our results we compare them with independent numerical computations by spectral method (Karpfinger, Gurevich and Bakulin 2007).

$$\boxed{\begin{array}{c}\langle S\rangle\\ h\ll r_0\end{array}} \xrightarrow[n=const]{\text{increase } h} \boxed{\langle S\rangle+\Delta} \xrightarrow[n\to\infty]{h=const} \boxed{\begin{array}{c}S_{aniso}\\ h\gg r\end{array}}$$

Figure 4 Evolution of the structure for propagator matrix G with total thickness h and number of periods n; Δ denotes all correction terms.

These computations lead to the following conclusions:

- both tube and plate wave velocities are almost constant for large numbers of periods (Figs 1 and 2);
- this limit can be described by simple volume averaging of material parameters (equation (A29)) if the curvature of the pipe is small ($h/r_0 \ll 1$) - model A (see Table 2);
- this limit generally cannot be described by simple volume averaging of material parameters if the curvature of the pipe is not small - model B (Fig. 1); note that the relative error for plate wave velocity remains very small (Fig. 2);
- in the case where sublayers in the period have equal shear moduli, velocities at zero frequency depend only on the relative volume concentration of sublayers (and are thus independent of number of periods) - model C (Fig. 3).

The velocity limit for large or infinite n corresponds to the velocity in the model with homogeneous effective anisotropic pipe, which is obtained from a propagator described by equation (45). The equivalence between finely layered and anisotropic pipes is proved in Appendix A. For all models these limits are denoted as 'Anisotropic' in Table 2.

A thin pipe is always well described by simple averages irrespective of the number of layers. Thick pipe becomes well described by the anisotropic limit for the large number of periods. In the intermediate case, when number of periods is not large while the pipe is thick, neither average nor anisotropic propagators provide a good description and matrix multiplication with correction terms (equation (A17)) should be carried out to obtain accurate results. We illustrate this behaviour by the diagram in Fig. 4.

CONCLUSIONS

We proposed a general form of the dispersion equation valid for low-frequency propagation of symmetric modes in fluid-filled boreholes and pipes consisting of radial elastic layers and having vacuum on the outside. This equation is expressed through the elements of a propagator matrix for axisymmetric waves characterizing the entire layered composite. We presented an exact and approximate solutions of the dispersion equations in the limit of zero frequency. We showed that previously known results are obtained as special cases of our general solutions.

We further presented propagator matrices for homogenous isotropic, layered and radially transversely isotropic pipes. We showed that a finely layered periodic medium made of isotropic cylindrical layers is equivalent to a homogeneous anisotropic media of transverse radial anisotropy. Numerical computations validate the good accuracy of the obtained equations for velocities of symmetric modes and confirm that layered pipes can be approximated by simple effective models. Theoretically, this replacement becomes exact when the number of periods increases to infinity. For practical applications we observe that an equivalent anisotropic media provides an excellent description, even for a relatively small number of periods and large acoustic contrast between the layers.

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REFERENCES

Achenbach J.D. 1970. Field equations governing the mechanical behavior of layered circular cylinders. *AIAA Journal* **8**, 1445–1452.

Bakulin A., Sidorov A., Kashtan B. and Jaaskelainen M. 2008. Real-time completion monitoring with acoustic waves. *Geophysics* **73**, E15–E33.

Backus G.E. 1962. Long-wave elastic anisotropy produced by horizontal layering. *Journal of Geophysical Research* **67**, 4427–4440.

Del Grosso V.A. 1971. Analysis of multimode acoustic propagation in liquid cylinders with realistic boundary conditions application to sound speed and absorbtion measurements. *Acustica* **24**, 299–311.

Ewing W.M., Jardetzky W.S. and Press F. 1957. *Elastic Waves in Layered Media*. McGraw-Hill. ISBN 0070198608.

Hill R. 1963. Elastic properties of reinforced solids: some theoretical principles. *Journal of the Mechanics and Physics of Solids* **11**, 357–372.

Karpfinger F., Gurevich B. and Bakulin A. 2007. Computing borehole modes with spectral method. 77th SEG meeting, San Antonio, Texas, USA, Expanded Abstracts, 333–337.

Lafleur L.D. and Shields F.D. 1995. Low-frequency propagation modes in a liquid-filled elastic tube waveguide. *Journal of the Acoustic Society of America* **97**, 1437.

Love A.E.H. 1944. *A Treatise on the Mathematical Theory of Elasticity*. Dover, New York. ISBN 0486601749.

Norris A.N. 1990. The speed of a tube wave. *Journal of the Acoustic Society of America* **87**, 414–417.

Petrashen G., Molotkov L. and Krauklis P. 1985. *Waves in Layered Isotropic Elastic Media II*. Nauka, Leningrad, Russia (in Russian).

White J.E. 1983. *Underground Sound*. Elsevier. ISBN 0444421394.

APPENDIX A: PROPAGATOR MATRIX FOR LAYERED ISOTROPIC MATERIAL

This appendix is devoted to detailed analysis of the propagator matrix of layered isotropic pipe in the low-frequency limit. In contrast to the plane-layered case, cylindrical layers have a second scale in addition to wavelength. This scale is a curvature of the pipe, defined as ratio of thickness h to the inner radius r_0 of the pipe. Whenever we refer to a 'thin' pipe we assume that $h \ll r_0$. Likewise, we refer to a 'thick' pipe in a more general case when h is comparable to or larger than r_0.

We also consider periodical structures and the limiting case of infinite number of periods with fixed overall thickness. Such a system is equivalent to the radially transversely isotropic homogeneous media suggested by Achenbach (1970). It corresponds to known results of equivalence of finely layered materials and transversely isotropic media (Backus 1962).

In the following we briefly show the main steps in obtaining the propagator matrix for cylindrical periodical structures and compare it with the matrix of transversely isotropic materials. After that we consider in detail propagator matrices for the layered pipe with a finite number of layers and periodical pipe with an infinite number of periods.

Radial anisotropy and periodical structure

Let us show that periodically layered pipe of a fixed thickness with an infinite number of thin periods can be described by a propagator matrix of radial transversely isotropic media described by equation (45) with appropriately chosen parameters c_{ij}. First we outline the scheme and state the results. Then we explain each step of the derivation.

This equivalence can be proved in three steps as illustrated by diagram in Fig. 5. First, we find the propagator matrix for a single infinitely thin period ($h/r_0 \to 0$), that is equivalent to thin homogeneous anisotropic pipe with anisotropic parameters c_{ij} given by Backus averaging (Backus 1962). This is shown on the left part of the diagram. Second, we build a propagator matrix of thick periodical pipe from the matrix of a single infinitely thin period. This corresponds to transferring from the left to the right part of the diagram. Third,

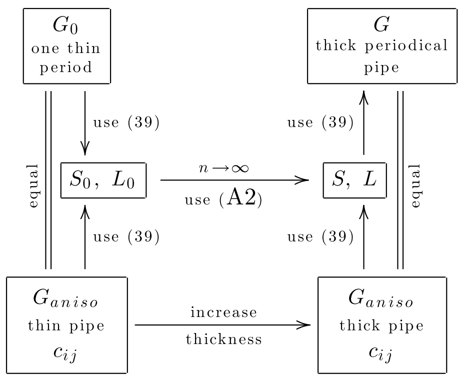

Figure 5 Diagram illustrating derivation scheme and relationships between various propagators.

we establish the equivalence between thick periodical pipe in the limit $n \to \infty$ and thick anisotropic pipe having parameters c_{ij} expressed by the same Backus averaging (Backus 1962) previously found for thin anisotropic pipe.

Defining periodical pipe, we assume that periods consist of homogeneous isotropic sublayers with constant volume concentration and the thicknesses of the periods are of the same order.

The left part of the diagram in Fig. 5 shows that for a given propagator matrix G_0 of infinitely thin period, we construct matrix S_0 of elements $s_{0\,ij}$ and vector L_0 of elements $l_{0\,1j}$ using representation equation (39). We do not mention elements $l_{0\,i4}$ because the fourth column of matrix G can be directly reconstructed using symmetry relationships equation (45b) applicable for both periodical and anisotropic pipes. When the period is thin, property matrices S_0 and L_0 are simply volume averaged matrices of sublayers. We then establish appropriate coefficients c_{ij} as functions of averages $s_{0\,ij}$ and thus obtain equivalent anisotropic thin pipe (see later equation (A31)). Note that for small thickness of the pipe (small y) we have $y_\pm \simeq \frac{\nu \pm 1}{2} y$ and thus the same geometric factors appear in periodical and anisotropic pipes.

Since we assumed constant volume concentration of sublayers and matrices S_0 and L_0 are volume averages, then they are the same for all infinitely thin periods of the pipe. Following the middle row of the diagram in Fig. 5, we should express matrices S and L of thick periodical pipe through matrices S_0 and L_0 of its periods. In order to do that, we introduce diagonal matrix $P = diag\{0 \quad 1 \quad 1 \quad 0\}$. Multiplication by P on the right-hand side replaces the first and the fourth columns of any 4×4 matrix by zeros. We show that to find a propagator of periodic pipe it is sufficient to derive expressions for a reduced matrix $Z = SP$, element s_{41} and property vector L. Property vector L' and other elements of property matrix S can be recovered from symmetry relationships equation (45b) and thus the entire propagator can be found. Let us define a matrix

$$Y = Y(r, r_0) = \left(\frac{r}{r_0}\right)^{2Z_0} - I, \tag{A1}$$

where $Z_0 = S_0 P$. As it will be shown by equation (A6a), parameters y_+ and y_- introduced in equation (44) are eigenvalues of Y. Then desired matrices required to define a periodic propagator in the limit $n \to \infty$ are expressed as

$$Z = \frac{1}{y} Y, \qquad L \ln \frac{r}{r_0} = L_0 \int_{r_0}^{r} \frac{dr'}{r'} [Y(r', r_0) + I], \tag{A2a}$$

$$\begin{aligned} s_{41} &= \langle s_{41} \rangle + k_{42} \langle s_{21} \rangle + k_{43} \langle s_{31} \rangle, \\ \text{where} \qquad K &= \left(Z_0 - \frac{1}{y} Y\right)(I - Z_0)^{-1} \end{aligned} \tag{A2b}$$

and $\langle s_{ij} \rangle$ are elements of matrix S_0 that represent volume averaging of the corresponding constituent layers in the period. These equations correspond to equations (A40), (A52) and (A37) considered later.

In the third step, shown as the right-hand column of the diagram on Fig. 5, we again use template (equation (39) to find propagator matrix G that simultaneously describes periodical pipe at the limit $n \to \infty$ and equivalent anisotropic pipe. Note that in the general case of a thick pipe, elements of S and L include complex geometry-property factors $y_\pm$ defined by equation (44). This manifests change of the propagator structure where clear decomposition in property and geometric factors is no longer possible. The section 'Numerical examples' illustrates this change of structure using diagram in Fig. 4 and numerical results shown in Figs 1 and 2.

Computation of effective material parameters

Let us describe how to calculate the matrix expressions (A2) postulated above. These expressions are functions of the reduced property matrix $Z_0 = S_0 P$ and can be calculated if matrix Z_0 is diagonalized. As will be shown in section 'Effective homogeneous media describing layered pipe of small thickness', elements of Z_0 can be considered as combinations of constants c_{ij} of some radial transversely isotropic material. After introducing parameter $\nu^2 = \frac{c_{33}}{c_{11}}$ and computing eigenvalues, matrix Z_0 can be diagonalized as

$$Z_0 = M\, diag\{0 \quad \tfrac{1+\nu}{2} \quad \tfrac{1-\nu}{2} \quad 0\}\, M^{-1}, \tag{A3}$$

where matrix M is formed by eigenvectors of Z_0:

$$M = \begin{bmatrix} 1 & \frac{1-\nu+\frac{1}{c_{44}}(\nu c_{11}+c_{13})}{(1+\nu)c_{11}} & m_{13} & 0 \\ 0 & \frac{(\nu c_{11}+c_{13})}{2c_{11}} & m_{23} & 0 \\ 0 & \frac{1}{c_{11}} & m_{33} & 0 \\ 0 & \frac{c_{23}+\nu c_{13}}{(1+\nu)c_{11}} & m_{43} & 1 \end{bmatrix}. \tag{A4}$$

Elements m_{j3} can be expressed through m_{j2} as

$$m_{j3} = \frac{1}{2}(\nu c_{11} + c_{13}) m_{j2}(-\nu). \tag{A5}$$

Expressions for eigenvalues of matrix Z_0 will be proved in section 'Eigenvalues of reduced property matrix'.

We observe that eigenvalues and eigenvectors only depend on anisotropic parameters c_{ij} and do not depend on geometric parameters. This also means that the structure of the anisotropic propagator for thin pipe is the same as for isotropic or layered case (equation (39)). This is a consequence of the fact that Z_0 describes an infinitely thin period.

Using equation (A3) we can rewrite equation (A2) in a diagonalized form as

$$yZ = M\, diag\{0 \quad y_+ \quad y_- \quad 0\}\, M^{-1}, \tag{A6a}$$

$$L \ln\frac{r}{r_0} = L_0 M\, diag\left\{\ln\frac{r}{r_0} \quad \frac{y_+}{1+\nu} \quad \frac{y_-}{1-\nu} \quad \ln\frac{r}{r_0}\right\} M^{-1}, \tag{A6b}$$

$$K = \frac{1}{y} M\, diag\left\{0 \quad \frac{1+\nu}{1-\nu}\left[y - \frac{2y_+}{1+\nu}\right] \quad \frac{1-\nu}{1+\nu}\left[y - \frac{2y_-}{1-\nu}\right] \quad 0\right\} M^{-1}. \tag{A6c}$$

We observe that geometry-property factors y_+ and y_- of thick anisotropic pipe appeared in eigenvalues corresponding to periodical pipe with an infinite number of periods.

To make it obvious that after substituting equations above into the form (equation (39)) we obtain exactly the propagator matrix (equation (45)) of anisotropic pipe, let us give a more detailed expression for the newly appeared quantity

$$D = M\, diag\{d_1 \quad d_2 \quad d_3 \quad d_4\}\, M^{-1}, \tag{A7}$$

which is contained in all expressions (equation (A6)). Elements of such a matrix product are

$$d_{jk} = \frac{(-1)^k}{\det M}[d_2 m_{j2} m_{2k} - d_3 m_{j3} m_{3k}], \quad j = 2, 3, \quad k = 2, 3; \tag{A8a}$$

$$d_{jk} = \frac{(-1)^k}{\det M}[(d_2 - d_j) m_{j2} m_{2k} - (d_3 - d_j) m_{j3} m_{3k}], \quad j = 1, 4, \quad k = 2, 3; \tag{A8b}$$

$$d_{jj} = d_j, \qquad j = 1, 4, \tag{A8c}$$

$$\text{where} \qquad \det M = \frac{\nu(\nu c_{11} + c_{13})}{2c_{11}}. \tag{A8d}$$

In conclusion, we should mention how to obtain propagators for the special case of $\nu = 1$. This may occur for isotropic or transversely isotropic material with $c_{23} \neq c_{13}$. In such a case, as one can see from matrix (A4), elements m_{13} and m_{43} tend to infinity proportionally to $(1-\nu)^{-1}$. However, these elements appear only in equation (A8b), where they are multiplied by $d_3 - d_4 \sim 1 - \nu$ (we always have $d_1 = d_4$ in our equations). Indeed, in all possible cases (for matrices S, L and K correspondingly) this uncertainty can be resolved utilizing following relationships:

$$y_- \sim (1-\nu)\ln\frac{r}{r_0}, \quad \frac{y_-}{1-\nu} - \ln\frac{r}{r_0} \sim \frac{1-\nu}{2}\ln\frac{r}{r_0}, \quad \frac{1-\nu}{1+\nu}\left[y - \frac{2y_-}{1-\nu}\right] \sim \frac{1-\nu}{2}[y - \ln(1+y)]. \tag{A9}$$

Multilayered systems

In this section we analyse the structure of the propagator matrix G, defined by equation (38). Representing matrix G in the form (39), we consider how its property-factors can be expressed through property-factors of component layers.

To construct such a representation, we notice that the structure of the diagonal elements of propagators (equations (27) and (39a)) naturally allows them to be represented as $G = I + \mathcal{G}$. We then can write equation (38) for $n = 2$ in the form

$$I + \mathcal{G}(r_2, r_0) = [I + \mathcal{G}_2(r_2, r_1)][I + \mathcal{G}_1(r_1, r_0)] \tag{A10}$$

and express $\mathcal{G}$ through $\mathcal{G}_2$ and $\mathcal{G}_1$ as

$$\mathcal{G}(r_2, r_0) = \mathcal{G}_2(r_2, r_1) + \mathcal{G}_1(r_1, r_0) + \mathcal{G}_2(r_2, r_1)\mathcal{G}_1(r_1, r_0).$$

There are two important observations concerning the equation above. The first is that it is an exact equation for matrices with all terms with ξ-expansion included. Since we are only interested in a low-wavenumber limit, we only consider the first term of each element. If we keep them for matrices $\mathcal{G}_2$ and $\mathcal{G}_1$, then higher-order terms naturally appear in the product $\mathcal{G}_2\mathcal{G}_1$ and we should neglect them. This could be formalized using projector matrices. In particular, three corner elements of $\mathcal{G}_2\mathcal{G}_1$ begin with the ξ^2-term, while in matrices $\mathcal{G}_j$ they begin with the ξ^0-term. To take it into account we introduce a linear operation $\{\cdot\}$ that acts on 4×4 matrices according to the rule: if $\hat{A} = \{A\}$ then $\hat{a}_{11} = \hat{a}_{14} = \hat{a}_{44} = 0$ and $\hat{a}_{ij} = a_{ij}$ for other elements. To avoid higher-order terms in the product, we should also put to zero elements $g_{24}^{(2)}, g_{34}^{(2)}$ of matrix $\mathcal{G}_2$ and $g_{12}^{(1)}, g_{13}^{(1)}$ of matrix $\mathcal{G}_1$. When we rewrite the equation for a layered propagator G in terms of property-factors, the described operation will be achieved by diagonal projector matrix $P = diag\,\{0\;1\;1\;0\}$.

A second observation is that the considered expression has the form of an arithmetic average $\mathcal{G}_2 + \mathcal{G}_1$, with some correction term $\mathcal{G}_2\mathcal{G}_1$. Therefore, we define averages $\langle\cdot\rangle$ and $\langle\cdot\rangle_{ln}$ with the following weights

$$\theta_i = \frac{r_i^2 - r_{i-1}^2}{r_n^2 - r_0^2}, \qquad \psi_i = \frac{\ln \frac{r_i}{r_{i-1}}}{\ln \frac{r_n}{r_0}}, \tag{A11}$$

where r_0 and r_n are the inner and outer radii of the pipe with n layers.

Let us gather property-factors s_{ij} into matrix S, l_{1j} into vector L and l_{j4} into vector L'. Matrices S_i and vectors L_i, L_i' of the property-factors describing the isotropic layer are given by

$$S_i = \begin{bmatrix} 0 & \frac{\lambda_i+\mu_i}{\mu_i(\lambda_i+2\mu_i)} & \frac{\lambda_i+\mu_i}{\lambda_i+2\mu_i} & 0 \\ \frac{\lambda_i\mu_i}{\lambda_i+2\mu_i} & \frac{\lambda_i+\mu_i}{\lambda_i+2\mu_i} & \frac{\mu_i(\lambda_i+\mu_i)}{\lambda_i+2\mu_i} & -\frac{\lambda_i+\mu_i}{\lambda_i+2\mu_i} \\ -\frac{\lambda_i}{\lambda_i+2\mu_i} & \frac{1}{\lambda_i+2\mu_i} & \frac{\mu_i}{\lambda_i+2\mu_i} & \frac{\lambda_i+\mu_i}{\mu_i(\lambda_i+2\mu_i)} \\ \frac{4\mu_i(\lambda_i+\mu_i)}{\lambda_i+2\mu_i} & \frac{\lambda_i}{\lambda_i+2\mu_i} & \frac{\lambda_i\mu_i}{\lambda_i+2\mu_i} & 0 \end{bmatrix}, \tag{A12a}$$

$$L_i = \begin{bmatrix} 0 & -\frac{\lambda_i+\mu_i}{\mu_i(\lambda_i+2\mu_i)} & \frac{\mu_i}{\lambda_i+2\mu_i} & \frac{1}{\mu_i} \end{bmatrix}, \tag{A12b}$$

$$L_i' = \begin{bmatrix} \frac{1}{\mu_i} & -\frac{\mu_i}{\lambda_i+2\mu_i} & -\frac{\lambda_i+\mu_i}{\mu_i(\lambda_i+2\mu_i)} & 0 \end{bmatrix}^t. \tag{A12c}$$

We are now ready to construct a low-frequency version for a two-layered propagator.

Proposition 3. A property matrix describing a pipe consisting of two layers with welded contact can be obtained as volum-averaged elements of constituent layers with some correction terms

$$S = \langle S\rangle + \frac{y_1 y_2}{y}\{S_2 P(S_1 - P)\}, \tag{A13a}$$

$$L = \langle L\rangle_{ln} + y_1\psi_2 L_2 S_1 P, \tag{A13b}$$

$$L' = \langle L'\rangle_{ln} + x_2\psi_1 P(S_2 - P)L_1', \tag{A13c}$$

where parameters

$$x_i = 1 - \frac{r_{i-1}^2}{r_i^2}, \quad y_i = \frac{r_i^2}{r_{i-1}^2} - 1, \quad y = \frac{r_n^2}{r_0^2} - 1 \tag{A14}$$

are used with $n = 2$ and $i = 1, 2$.

Layers in this proposition can be inhomogeneous and equation (A13) can be applied recursively to obtain S, L and L' for the multilayered case. For example, in the case of composite pipe consisting of three isotropic layers we obtain

$$\begin{aligned} S = \langle S\rangle &+ \frac{y_3 y_2}{y}\{S_3 P(S_2 - P)\} \\ &+ \frac{y_3 y_1}{y}\{S_3 P(S_1 - P)\} + \frac{y_2 y_1}{y}\{S_2 P(S_1 - P)\} \\ &+ \frac{y_3 y_2 y_1}{y}\{S_3 P(S_2 - I) + S_3 P S_2 P(S_1 - I)\}. \end{aligned} \tag{A15}$$

Recursively applying equation (A13a) we can derive the expression for n layers. They will also contain volume-averaged terms $\langle S\rangle$ and additional terms of progressively higher orders of y_i.

However, a more convenient generalization can be produced using the matrices of property-factors $S_{(n,k)}$ and geometric parameters $y_{(n,k)}$ computed for the group of layers with numbers $k, \dots, n$. For simplification we use the notation $S_{(k)} = S_{(k,1)}$ and $S_k = S_{(k,k)}$. Let us denote combinations involving the projector as

$$Z = SP \quad \text{and} \quad Z' = P - PS,$$

which can be similarly defined for matrices S and Z with indices. Volume-averaging weights can be expressed through geometric parameters as

$$\theta_i = \frac{y_i(1 + y_{i-1})\dots(1 + y_1)}{y}, \qquad \theta_1 = \frac{y_1}{y}. \tag{A16}$$

As a result, for the case of n layers we derive general expressions for property matrices and vectors defining a multilayered propagator

$$S = \langle S\rangle - \left\{\sum_{k=2}^{n} y_{(n,k)} Z_{(n,k)}\theta_{k-1} Z'_{k-1}\right\}, \tag{A17a}$$

$$L = \langle L\rangle_{ln} + \sum_{k=2}^{n} \psi_k L_k y_{(k-1)} Z_{(k-1)}, \tag{A17b}$$

$$L' = \langle L'\rangle_{ln} - \sum_{k=2}^{n} x_{(n,k)} Z'_{(n,k)}\psi_{k-1} L'_{k-1}. \tag{A17c}$$

There exists a simple representations for matrices $Z_{(n,k)}$ and $Z'_{(n,k)}$ used in the above expressions. Multiplying equation (A13a) by P on either righthand or lefthand side, and using property $\{A\}\,P = AP$ or $P\{A\} = PA$ and expressions (A16), we reduce equation (A13a) to the following form

$$I + yZ = (I + y_2 Z_2)(I + y_1 Z_1), \qquad \text{or} \tag{A18a}$$

$$I - xZ' = (I - x_2 Z_2')(I - x_1 Z_1'), \tag{A18b}$$

which can be generalized for the group of n layers as

$$I + y_{(n,k)} Z_{(n,k)} = (I + y_n Z_n) \ldots (I + y_k Z_k), \tag{A19a}$$

$$I - x_{(n,k)} Z'_{(n,k)} = (I - x_n Z'_n) \ldots (I - x_k Z'_k). \tag{A19b}$$

Parameters $y_{(n,k)}$ and $x_{(n,k)}$ can be similarly rewritten as

$$1 + y_{(n,k)} = (1 + y_n) \ldots (1 + y_{k+1})(1 + y_k), \tag{A20a}$$

$$1 - x_{(n,k)} = (1 - x_n) \ldots (1 - x_{k+1})(1 - x_k). \tag{A20b}$$

Each set of geometric parameters x or y can be used and simple relationships exist between them for the case of n layers

$$y_i = \frac{x_i}{1 - x_i}, \qquad \frac{y_n \ldots y_1}{y} = \frac{x_n \ldots x_1}{x}. \tag{A21}$$

Note that order of matrices in the products above is fixed. The presence of correction terms does not allow layers to be interchanged. This in contrast to a medium of fine plane layers where layers can be interchanged at low frequencies and the propagator only contains volume-averaged terms. We will further observe how correction terms accumulate for large n and transform the structure of the propagator to the new type.

Analysis of correction terms

Let us analyse the correction term, which is the difference between matrix S of layered material and averaged matrix $\langle S \rangle$. *The difference $S - \langle S \rangle$ is controlled by curvature $\tau = h/r_0$ and shear modulus difference $\Delta\,\mu = \max\,(\mu_i - \mu_j)$.* It is convenient to use usual parameter y expressed through τ as $y = \tau(2 + \tau)$. For any number of layers this difference is bounded by

$$\| S - \langle S \rangle \| \leq \left[1 - \frac{\ln(1 + y)}{y}\right] \Delta\mu D, \tag{A22}$$

where D depends only on the material parameters of constituent layers. For small y we can write a more crude estimate

$$\| S - \langle S \rangle \| \leq \frac{y}{2} \Delta\mu D \simeq \tau \Delta\mu D. \tag{A23}$$

To explain the origin of this estimate, let us begin with the geometrical part. Generalizing equation (A15) to n layers we can write

$$\begin{aligned} \| S - \langle S \rangle \| &\leq \left[\sum_{1 \leq j < i \leq n} \frac{y_i y_j}{y} + \ldots + \frac{y_n \ldots y_1}{y}\right] C \\ &= \frac{1}{y}\left[(1 + y) - \left(1 + \sum_{k=1}^{n} y_k\right)\right] C, \end{aligned} \tag{A24}$$

where we used representation (A20a) for y; constant C is independent of radii. We then note that function $f(y_n, \ldots, y_1) = \sum_{k=1}^{n} y_k$ has its conditional minimum when all y_k are equal to some fixed value y_0. Relation (A20a) gives $(1 + y_0)^n = 1 + y$ and we further simplify to

$$\begin{aligned} \| S - \langle S \rangle \| &\leq \frac{1}{y}\left[y - \sum_{k=1}^{n} y_k\right] C \leq 1 - \frac{n y_0}{y} \\ &= \left[1 - \frac{y_0}{y} \frac{\ln(1 + y)}{\ln(1 + y_0)}\right] C \leq \left[1 - \frac{\ln(1 + y)}{y}\right] C. \end{aligned} \tag{A25}$$

To illustrate the dependence on $\Delta\mu$ we start with the case of $n = 2$. First, we obtain a more specific expression derived from Eq. (A31a) and describing two-component pipe with arbitrary first layer but isotropic second layer

$$S = \langle S \rangle - \frac{y_1 y_2}{y} \{S_2 P \Delta S\}, \tag{A26}$$

where $\Delta S = S_2 - S_1$. This can be verified by reminding you that the property of a propagator for an isotropic homogeneous layer $\{S_2 P\,(S_2 - P)\} = 0$, which in turn can be obtained from equation (A13a) by considering the case of bilayered pipe with identical layers.

When both layers are homogeneous and isotropic then we can obtain an even more detailed result. Isotropic property matrices from equation (A13a) possess the following symmetry property: the second and the third columns of matrix S_i are dependent as well as two middle rows of matrix $S_i - P$. If we denote the second column of S_2 as $\boldsymbol{c}_2$, and the third row of $S_1 - P$ as $\boldsymbol{t}_1$, then

$$S_2 P = \{0 \quad \boldsymbol{c}_2 \quad \mu_2 \boldsymbol{c}_2 \quad 0\}, \qquad P S_1 - P = \begin{Bmatrix} 0 \\ -\mu_1 \boldsymbol{t}_1 \\ \boldsymbol{t}_1 \\ 0 \end{Bmatrix},$$

and $S_2 P\,(S_1 - P) = (\mu_2 - \mu_1)\,\boldsymbol{c}_2\,\boldsymbol{t}_1$. Therefore, the correction term is proportional to the difference between shear moduli and equation (A13a) now turns into

$$S = \langle S \rangle - \frac{y_2 y_1}{y} \Delta\mu \{\mathcal{D}\}, \tag{A27}$$

where $\mathcal{D} = -\boldsymbol{c}_2 \boldsymbol{t}_1$ and $\Delta\mu = \mu_2 - \mu_1$.

This result can be generalized for the case of three layers after noticing that the last term of equation (A15) can be written in the form

$$\begin{aligned} &S_3 P (S_2 - P) + S_3 P S_2 P (S_1 - P) \\ &\qquad = -\Delta\mu_{32} \mathcal{D}_{32} - \Delta\mu_{21} S_3 P \mathcal{D}_{21}, \end{aligned} \tag{A28}$$

where $\Delta\mu_{32} = \mu_3 - \mu_2$ and $\Delta\mu_{21} = \mu_2 - \mu_1$.

Effective homogeneous media describing layered pipe of small thickness

When pipe is thin (has a small curvature $\tau \ll 1$), then correction terms in equation (A17) can be neglected in S, L and L' and we arrive at a simpler expressions

$$S \simeq \langle S\rangle, \qquad L \simeq \langle L\rangle, \qquad L' \simeq \langle L'\rangle. \tag{A29}$$

Note that we are allowed to use only volume average, because all weights (A11) become identical $\theta_i \simeq \psi_i \simeq h_i/h$ for thin pipe. Therefore, we arrived at the following result: *if layered pipe is thin compared to its inner radius ($\tau \ll 1$), then it can be replaced by homogeneous radially anisotropic pipe with five elastic parameters*. Such materials were called radial transverse isotropy by Love (1944).

Convergence of volume and logarithmic averaging for thin pipe preserves the same relationships (equation (A12)) between elements of L, L' and S as for isotropic layers

$$l_{12} = -s_{12}, \qquad l_{13} = 1 - s_{13}, \qquad l_{14} = s_{12} + s_{32},$$
$$l_{34} = -s_{34}, \qquad l_{24} = -1 - s_{24}.$$

For thirteen elements s_{ij} there also exist eight additional relationships

$$\begin{aligned} &s_{21} = s_{43}, && s_{22} = s_{13}, && s_{24} = -s_{13},\\ &s_{31} = -s_{42}, && s_{33} = 1 - s_{22}, && s_{34} = s_{12},\\ &s_{41} = 4s_{23}, && 2s_{22} = 1 + s_{42}. \end{aligned} \tag{A30}$$

Therefore, only five property constants remain independent in the layered propagator matrix G. Another material, characterized by five independent material constants, is radial transverse isotropy with the stress-strain law given by equation (43). In order to replace the considered thin packet of homogeneous isotropic layers with anisotropic homogeneous pipe, we should relate their parameters as follows

$$\begin{aligned} &c_{11} = \frac{1}{\langle s_{32}\rangle}, \quad c_{33} = 4\langle s_{23}\rangle + \frac{\langle s_{42}\rangle^2}{\langle s_{32}\rangle}, \quad c_{44} = \frac{1}{\langle l_{14}\rangle},\\ &c_{13} = \frac{\langle s_{42}\rangle}{\langle s_{32}\rangle}, \quad c_{23} = 2\langle s_{43}\rangle + \frac{\langle s_{42}\rangle^2}{\langle s_{32}\rangle}, \end{aligned} \tag{A31}$$

where $\langle s_{ij}\rangle$ denotes volume averaging of corresponding elements of the property matrix for thin stack of layers. The expressions (A31) are identical to the well-known Backus (1962) averaging for effective anisotropic media replacing stack of isotropic plane layers. This is also in agreement with the results of Achenbach (1970) for cylindrical layers obtained by different means.

It can easily be verified that if thin pipe ($\tau \ll 1$) consists of homogeneous isotropic layers with equal shear moduli, then the effective medium is isotropic with the same μ whereas the effective λ is given by

$$\frac{1}{\lambda + 2\mu} = \sum_{m=1}^{n} \theta_m \frac{1}{\lambda_m + 2\mu}. \tag{A32}$$

It is consistent with general results of Hill (Hill 1963) for microinhomogeneous media. Note, that $S = \langle S\rangle$ even for thick pipe, but correction terms for L and L' do not vanish when λ_i are not equal for all layers. Thus, we need small curvature to use all of equation (A29).

Eigenvalues of a reduced property matrix

In the next section we consider a propagator for periodically layered pipe where it is convenient to operate with diagonalized property matrices. Thus, we need to determine the eigenvalues of the property matrix especially for the case of infinitely thin layered pipe, described in previous section 'Effective homogeneous media describing layered pipe of small thickness'.

Consider reduced property matrix $Z = Z_{(n)}$ of infinitely thin pipe made of n isotropic elastic layers. The first and the fourth columns are zero in the matrices of layers Z_k as well as in the matrix Z for the stack of n layers. Therefore, they all have two zero eigenvalues. The remaining two eigenvalues can be determined if both Tr Z and $\det(Z - I)$ are known (note that always $\det Z = 0$).

Now, let us calculate nonzero eigenvalues z_2 and z_3 for the case of the infinitely thin pipe (A12a) reveals that the trace of the property matrix S for a single isotropic layer as well as for a volume-averaged stack of layers $< S >$, is equal to unity

$$\mathrm{Tr} Z = z_2 + z_3 = 1. \tag{A33}$$

To compute the determinant we use relationships equation (A30) and expression (A31) and arrive to

$$\begin{aligned} \det(Z - I) &= s_{22}s_{33} - s_{23}s_{32}\\ &= \frac{1 - s_{42}^2}{4} - s_{23}s_{32} = \frac{1}{4}\left(1 - \frac{c_{33}}{c_{11}}\right), \end{aligned} \tag{A34}$$

where $s_{ij} = \langle s_{ij}\rangle$. Introducing parameter $\nu^2 = c_{33}/c_{11}$, we obtain the eigenvalues of reduced property matrix Z describing infinitely thin pipe as

$$z_2 = \frac{1 + \nu}{2}, \qquad z_3 = \frac{1 - \nu}{2}. \tag{A35}$$

It can be shown that $\nu = 1$ in both cases of constant μ or $\lambda + \mu$.

Periodically layered pipe

In this section we consider the application of the previous results (A17) and (A19) for periodically layered pipe of arbitrary total thickness. As shown in section 'Effective homogeneous media describing layered pipe of small thickness', at low frequencies thin pipe ($h/r_0 \ll 1$) can be replaced by effective homogeneous pipe with radial transverse isotropy. Here we generalize this result and show that even for a large total thickness, pipe formed by an infinite number of thin periods can be replaced with anisotropic homogeneous pipe characterized by the material parameters of a single averaged thin period. The propagator matrix for such anisotropic pipe was already presented by equation (45).

In order to find the propagator matrix of such a periodical structure we need to compute the limits

$$S = \lim_{n\to\infty} S_{(n)}, \quad L = \lim_{n\to\infty} L_{(n)}, \quad L' = \lim_{n\to\infty} L'_{(n)}.$$

It is natural to assume that the volume concentration of all constituents is constant for all periods. However, in order to compute the limits, one also needs to say something about period geometries. For cylindrical layers one has multiple choices, for example, to consider periods of constant thickness or constant area. For averaging purposes, it is advantageous to start with a special case when y_k for all periods is equal to each other. This case is important because the property matrices and property vectors for each period are also equal, which gives the shortest path to the desired limit. For example, equation (A13a) confirms that for equal volume concentrations the first terms ($< S>$) are the same for all periods. If all y_k are equal for all periods, then the correction terms are also equal for all periods (note that y_k corresponds to y in equation (A13a). Therefore the propagator matrices are all equal.

Calculation of property matrix S for an infinite number of periods

As a first step we calculate this limit for a reduced property matrix Z, in the simple case when all y_k are equal to some $y_0(n)$ and all Z_k are equal to some $Z_0(n)$

$$I + yZ = \lim_{n\to\infty} (I + y_0(n) Z_0(n))^n, \tag{A36}$$

where $y_0(n)$ is found from the equation $1 + y = (1 + y_0)^n$. As a second step, we show that periodic pipes with different period geometries, which satisfy condition (A45), also lead to the same limit.

Once the limit for the reduced matrix Z is found, the full property matrix S can be recovered using the expression

$$S - S_0 = \{(Z - Z_0)(I - Z_0)^{-1}(P - S_0)\}. \tag{A37}$$

To obtain this relation we first need to recall that $S_0 = \langle S\rangle$, $Z_0 = \lim_{n\to\infty} Z_0(n) = \langle Z\rangle$. Second, we observe that equation (A17a) can be written as $S - S_0 = \{\Sigma_n(S_0 - P)\}$. Multiplying this expression by P on the right-hand side, we obtain $Z - Z_0 = \Sigma_n(Z_0 - I)$, which allows one to express $\Sigma_n = (Z - Z_0)(Z_0 - I)^{-1}$. As a result, the property matrix S of the entire periodical structure can be expressed by equation (A37) through Z, Z_0 and S_0.

In the remainder of this appendix we will prove the equivalence by showing that for a thick pipe of finite thickness made of infinitely thin periods ($\tau \to 0$) we have

$$I + yZ = (1 + y)^{Z_0}. \tag{A38}$$

This equation can be substituted into equation (A37) and represents previously stated result in equation (A2b). Note that if the uncertainty appears due to $\det(Z_0 - I) = 0$, then it can always be resolved by means similar to example in equation (A9).

As we deal with functions of matrix Z_0, it is convenient to perform diagonalization and consider all expressions in terms of eigenvalues. Let us denote the elected eigenvalue of matrix Z_0 as Z_0, and the corresponding eigenvalue of matrix Z as z. Equation (A20a) gives $1 + y = (1 + y_0)^n$ and we find that

$$ny_0 = \frac{y_0 \ln(1+y)}{\ln(1+y_0)} \underset{[n\to\infty]}{\longrightarrow} \ln(1+y). \tag{A39}$$

We then rewrite equation (A36) for single eigenvalue z_0 and z as

$$1 + yz = \lim_{n\to\infty} e^{n\ln(1+y_0 z_0)} = e^{z_0 \ln(1+y)} = (1+y)^{z_0}. \tag{A40}$$

It remains to prove that the established limit remains the same for other period geometries with non-constant y_k and non-constant Z_k. Thus, we need to prove that matrix (A19a) made of factors $(I + y_k Z_k)$ and a matrix made of factors $(I + y_0 Z_0)$ are close to each other and become identical when $n \to \infty$. We accomplish this proof in two steps. In the first step, we estimate the error of replacing matrix (A19a) made of $(I + y_k Z_k)$ with a matrix made of $(I + y_k Z_0)$. In the second step, we estimate the error of replacing a matrix made of $(I + y_k Z_0)$ with a matrix made of $(I + y_0 Z_0)$.

Let us perform the first step and replace the product (A19a) by a similar product with $Z_k = Z_0$

$$I + y\widetilde{Z}_{(n)} = (I + y_n Z_0)\ldots(I + y_2 Z_0)(I + y_1 Z_0), \tag{A41}$$

and estimate the error introduced by such a replacement. It follows from equation (A23) that

$$\| Z_k - Z_0 \| < C \max_k y_k = C\delta, \tag{A42}$$

where C is some constant. For a matrix of any period we can write

$$I + y_k Z_k = y_k(Z_k - Z_0) + (I + y_k Z_0). \tag{A43}$$

After transformations

$$\begin{aligned} I + yZ_{(n)} &= (I + y_n Z_n)\ldots(I + y_2 Z_2)y_1(Z_1 - Z_0) \\ &+ \ldots + y_n(Z_n - Z_0)(I + y_{n-1}Z_0)\ldots(I + y_1 Z_0) \\ &+ (I + y_n Z_0)\ldots(I + y_2 Z_0)(I + y_1 Z_0), \end{aligned}$$

we obtain following estimate

$$\begin{aligned} y \parallel Z_{(n)}) - \widetilde{Z}_{(n)} \parallel &< n \max_k y_k \parallel Z_k - Z_0 \parallel \\ &\times (\parallel 1 + y_k Z_0 \parallel + y_k \parallel Z_k - Z_0 \parallel)^{n-1} \\ &< Cn\delta^2(1 + C\delta^2 + \delta|Z_0|)^{n-1}, \end{aligned}$$

where $c_0 = \parallel Z_0 \parallel$. It is convenient but not restrictive to assume that $C > 1$. We then arrive at the final estimate of the error valid for sufficiently large n when $\delta < c_0$:

$$y \parallel Z_{(n)} - \widetilde{Z}_{(n)} \parallel < Cn\delta^2 e^{2c_0 n\delta}. \tag{A44}$$

Note that for any k we have $y_k \geq 2h_k/r = 2h/(rn)$ and thus $n\delta = n \max y_k \geq C_1$ (h and r are the thickness and outer radius of the pipe). However, the exponential term in the right-hand side remains finite only if $n\delta \leq C_2$, where C_2 is some constant. Thus, we obtain a condition on y_k that periodic media should satisfy in order to justify this first replacement:

$$\frac{C_1}{n} \leq y_k = \frac{C_k(n)}{n} \leq \frac{C_2}{n}, \quad \text{or} \quad y_k \sim \frac{1}{n}. \tag{A45}$$

Under this condition, replacement error becomes small and goes to zero when $n \to \infty$. The simple geometrical meaning of this condition is that period thicknesses should remain of the same order while $n \to \infty$.

Let us now make the second step and replace all y_k with y_0 determined by $(1 + y_0)^n = 1 + y$ and estimate the error of such a replacement. Note that $y_0 \simeq \frac{1}{n}\ln(1 + y)$ and thus meets the condition (A45). It is more convenient to deal with the sum of the matrices instead of the product. Once we replace all Z_k by Z_0, we can take logarithms on both sides of equation (A41). Using the general inequality $\ln q \leq q - 1$, we obtain for selected eigenvalue z_0 that

$$\ln\frac{1 + y_k z_0}{1 + y_0 z_0} \leq \frac{(y_k - y_0)z_0}{1 + y_0 z_0}. \tag{A46}$$

We then write for a finite number of periods:

$$\ln\frac{(1 + y_n z_0)\ldots(1 + y_1 z_0)}{(1 + y_0 z_0)^n} \leq \frac{z_0}{1 + y_0 z_0}\sum_{k=1}^{n}(y_k - y_0). \tag{A47}$$

To continue estimations we need to use general inequalities

$$\ln(1 + y_k) \geq y_k - \tfrac{y_k^2}{2}, \qquad y_0 \geq \ln(1 + y_0),$$

which are valid for $y_k > 0$, and obtain

$$\begin{aligned} \ln(1 + y) &= \sum_{k=1}^{n}\ln(1 + y_k) \geq \sum_{k=1}^{n}\left(y_k - \frac{y_k^2}{2}\right) \\ &= \sum_{k=1}^{n}(y_k - y_0) - \sum_{k=1}^{n}\frac{y_k^2}{2} + ny_0 \\ &\geq \sum_{k=1}^{n}(y_k - y_0) - \sum_{k=1}^{n}\frac{y_k^2}{2} + \ln(1 + y). \end{aligned} \tag{A48}$$

Cancelling out identical terms, we arrive at

$$\sum_{k=1}^{n}(y_k - y_0) \leq \sum_{k=1}^{n}\frac{y_k^2}{2}. \tag{A49}$$

Assuming the same condition $y_k \sim \frac{1}{n}$ is fulfilled, we can see that replacement error (equation (A47)) is indeed small

$$\ln\frac{(1 + y_n z_0)\ldots(1 + y_1 z_0)}{(1 + y_0 z_0)^n} \leq z_0 n \max_k \frac{y_k^2}{2} \sim \frac{1}{n} \to 0.$$

Thus, we conclude the proof that the limit (equation (A40)) does not depend on the details of y_k when they are of the same order, which is expressed by condition (A45).

Calculation of property vector L for an infinite number of periods

To obtain vector L we should find the limit of equation (A17b)

$$L_{(n)}\ln\frac{r}{r_0} = \sum_{k=0}^{n-1}L_{k+1}[I + y_{(k)}Z_{(k)}]\ln\frac{r_{k+1}}{r_k}. \tag{A50}$$

When $n \to \infty$ then the periods become thin and their property vectors converge $L_k \to \mathrm{L}_0$. For selected z_0, let us denote $z(r')$ as an eigenvalue of the intermediate layer bounded by r_0 and r' with an infinite number of periods. We also denote $y(r') = (\frac{r'}{r})^2 - 1$. We then utilize equation (A40) in the form

$$1 + y(r')z(r') = (1 + y(r'))^{z_0} = \left(\frac{r'}{r_0}\right)^{2z_0}. \tag{A51}$$

Under condition (A45) and for large n we can write $\ln\frac{r_{k+1}}{r_k} = \ln(1 + \frac{h_{k+1}}{r_k}) \simeq \frac{h_{k+1}}{r_k}$, which is replaced by $\frac{dr_k}{r_k}$ at the limit of infinite n. Taking all this into account, we obtain for the selected eigenvalue z_k of matrix $Z_{(k)}$ that

$$\sum_{k=0}^{n-1}[1+y_{(k)}z_k]\ln\frac{r_{k+1}}{r_k}\underset{[n\to\infty]}{\longrightarrow}\int_{r_0}^{r}\frac{dr'}{r'}[1+y(r')z(r')]$$
$$=\int_{r_0}^{r}\frac{dr'}{r'}\left(\frac{r'}{r_0}\right)^{2z_0}=\begin{cases}\frac{(\frac{r}{r_0})^{2z_0}-1}{2z_0}, & \text{if } z_0\neq 0,\\ \ln\frac{r}{r_0}, & \text{if } z_0=0.\end{cases} \tag{A52}$$

The eigenvalues of Z_0 are $\{0 \quad \frac{1+\nu}{2} \quad \frac{1-\nu}{2} \quad 0\}$, as shown in section 'Eigenvalues of reduced property matrix'. Then, property vector for periodically layered pipe is expressed as

$$L\ln\frac{r}{r_0}=L_0 M\, diag\left\{\ln\frac{r}{r_0} \quad \frac{y_+}{1+\nu} \quad \frac{y_-}{1-\nu} \quad \ln\frac{r}{r_0}\right\}M^{-1}. \tag{A53}$$

Note that additional geometry-property parameters of anisotropic material y_+ and y_- emerged after taking the limit (they are defined by equation (44)). This evolution of the propagator structure is depicted by the diagram in Fig. 4. We start with a single period with the simplest propagator structure, similar to the structure of isotropic homogeneous pipe. For a finite number of periods inside the thick pipe this structure becomes more complex because of the presence of correction terms due to curvature. After taking the limit ($n\to\infty$) we arrive at a completely different structure, identical to that of a thick anisotropic pipe. We emphasize that the new structure only emerges after accumulating an infinite number of corrections terms.

Finally we point out that equation (A40) coupled with equations (A37) and (A53) allows to calculate all elements of S and L and provide initially stated equation (A2).

APPENDIX B: DERIVATION OF PROPAGATOR MATRIX FOR AXISYMMETRIC WAVES IN ISOTROPIC ELASTIC MEDIA

We restrict our derivation to only axisymmetric modes. In cylindrical coordinates (r, ϕ, z) this means $u_\phi = 0$ and $\partial_\phi = 0$. Let us consider monochromatic waves propagating in an axial z-direction with amplitudes dependent on r

$$\begin{aligned}u_z(r,z;t)&=u_z(r)\sin(\omega t+\xi z),\\ u_r(r,z;t)&=u_r(r)\cos(\omega t+\xi z).\end{aligned} \tag{B1}$$

Assuming linearly elastic isotropic material, we can write the differential equations on amplitudes u_r and u_z as

$$\begin{aligned}&\rho\omega^2u_r-\mu\xi^2u_r+(\lambda+2\mu)\left[\partial_r^2u_r+\frac{\partial_r u_r}{r}-\frac{u_r}{r^2}\right]\\ &\quad+(\lambda+\mu)\xi\,\partial_r u_z=0,\\ &\rho\omega^2u_z-(\lambda+2\mu)\xi^2u_z+\mu\left[\partial_r^2u_z+\frac{\partial_r u_z}{r}\right]\\ &\quad-(\lambda+\mu)\xi\left[\partial_r u_r+\frac{u_r}{r}\right]=0.\end{aligned} \tag{B2}$$

In the following, we use parameters α and β defined as

$$\alpha^2=\left(\frac{\rho c^2}{\lambda+2\mu}-1\right)\xi^2,\quad \beta^2=\left(\frac{\rho c^2}{\mu}-1\right)\xi^2, \tag{B3}$$

where c is phase velocity $c=\frac{\omega}{\xi}<\infty$. Let us also define vectors

$$\boldsymbol{u}=\begin{bmatrix}u_r\\ u_z\end{bmatrix},\qquad \boldsymbol{\sigma}=\begin{bmatrix}\sigma_{rr}\\ \sigma_{rz}\end{bmatrix}. \tag{B4}$$

Equation (B2) has four independent solutions:

$$\boldsymbol{u}_1=\begin{bmatrix}\frac{\alpha}{\xi}J_1(\alpha r)\\ J_0(\alpha r)\end{bmatrix},\qquad \boldsymbol{u}_2=\begin{bmatrix}\frac{\alpha}{\xi}N_1(\alpha r)\\ N_0(\alpha r)\end{bmatrix}, \tag{B5}$$

$$\boldsymbol{u}_3=\begin{bmatrix}\frac{\xi}{\beta}J_1(\beta r)\\ -J_0(\beta r)\end{bmatrix},\qquad \boldsymbol{u}_4=\begin{bmatrix}\frac{\xi}{\beta}N_1(\beta r)\\ -N_0(\beta r)\end{bmatrix}, \tag{B6}$$

expressed through special functions J, N defined by

$$\begin{aligned}J_n(x)&=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\ldots,\\ \frac{\pi}{2}N_0(x)&=J_0(x)\ln\frac{x}{2}-\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\left[\sum_{n=1}^{m}\frac{1}{n}-C\right],\\ \frac{\pi}{2}N_1(x)&=J_1(x)\ln\frac{x}{2}-\frac{1}{x}+O(x).\end{aligned} \tag{B7}$$

Any common solution is a linear combination with arbitrary constants. This combination is the same for both displacements and stresses:

$$\begin{aligned}\boldsymbol{u}&=\boldsymbol{u}_1Z_1+\boldsymbol{u}_2Z_2+\boldsymbol{u}_3Z_3+\boldsymbol{u}_4Z_4,\\ \boldsymbol{\sigma}&=\boldsymbol{\sigma}_1Z_1+\boldsymbol{\sigma}_2Z_2+\boldsymbol{\sigma}_3Z_3+\boldsymbol{\sigma}_4Z_4.\end{aligned} \tag{B8}$$

It is convenient to introduce a vector

$$W=(u_z\sigma_{rr}ru_r r\sigma_{rz})^t \tag{B9}$$

and rewrite equation (B8) in matrix form $W(r) = M(r)\, Z$, where matrix M has elements u_{zi}, σ_{rri}, ru_{ri}, $r\sigma_{rzi}$ with $i = 1,\ldots,4$. On the other hand, for some different radius r_0 we have $W(r_0) = M(r_0)Z$. Thus, we can relate vectors $W(r)$ and $W(r_0)$ as

$$W(r)=M(r)M^{-1}(r_0)W(r_0)=G(r,r_0)W(r_0) \tag{B10}$$

via a propagator matrix $G(r, r_0)$. To establish the exact representation for the propagator matrix we closely follow a recipe proposed by Molotkov (Petrashen *et al.* 1985).

Proposition 4. Matrix $M(r)$ can be represented in the form of the following product

$$M(r) = F(r)AQS(r), \tag{B11}$$

where

$$F(r) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2\mu}{r^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ \frac{g}{\xi} & 2\xi\mu & 0 & 0 \\ 0 & 0 & \frac{1}{\xi} & \frac{\xi}{\beta^2} \\ 0 & 0 & -2\mu & \frac{g}{\beta^2} \end{bmatrix},$$

$$S(r) = \begin{bmatrix} J_0(\alpha r) & N_0(\alpha r) & 0 & 0 \\ \alpha r J_1(\alpha r) & \alpha r N_1(\alpha r) & 0 & 0 \\ 0 & 0 & J_0(\beta r) & N_0(\beta r) \\ 0 & 0 & \beta r J_1(\beta r) & \beta r N_1(\beta r) \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g = \rho\omega^2 - 2\mu\xi^2. \tag{B12}$$

Using this expansion of matrix $M(r)$ and $Q^{-1} = Q$ we can write

$$G(r, r_0) = F(r)AQS(r)S^{-1}(r_0)QA^{-1}F^{-1}(r_0). \tag{B13}$$

If we introduce functions

$$S_\alpha^{kl}(r, r_0) = -\frac{\pi}{2}(\alpha r)^k(\alpha r_0)^l[J_k(\alpha r)N_l(\alpha r_0) - N_k(\alpha r)J_l(\alpha r_0)], \tag{B14}$$

then we can write $S(r)S^{-1}\ (r_0)$

$$= \begin{bmatrix} S_\alpha^{01}(r, r_0) & S_\alpha^{00}(r_0, r) & 0 & 0 \\ S_\alpha^{11}(r, r_0) & S_\alpha^{01}(r_0, r) & 0 & 0 \\ 0 & 0 & S_\beta^{01}(r, r_0) & S_\beta^{00}(r_0, r) \\ 0 & 0 & S_\beta^{11}(r, r_0) & S_\beta^{01}(r_0, r) \end{bmatrix}. \tag{B15}$$

It is important to consider functions S^{ik} in detail using representations

$$\frac{\pi}{2}N_0(z) = J_0(z)\ln\frac{z}{2} - P_0(z), \tag{B16}$$

$$\frac{\pi}{2}N_1(z) = J_1(z)\ln\frac{z}{2} - \frac{1}{z} - zP_1(z), \tag{B17}$$

where $P_0(z)$ and $P_1(z)$ are some polynomials of z^2. Then

$$S_\alpha^{01}(r, r_0) = J_0(\alpha r) + \alpha r_0 J_0(\alpha r)J_1(\alpha r_0)\ln\frac{r}{r_0} - \alpha r_0 P_0(\alpha r)J_1(\alpha r_0) + (\alpha r_0)^2 J_0(\alpha r)P_1(\alpha r_0). \tag{B18}$$

We can consider the last expression as function of ξ^2 and expand it as a series using equation (B3)

$$S_\alpha^{01}(r, r_0) = 1 + s'_{01}\xi^2 + s''_{01}\xi^4 + \dots. \tag{B19a}$$

In the same way we may expand other elements

$$S_\alpha^{00}(r_0, r) = s_{00} + s'_{00}\xi^2 + s''_{00}\xi^4 + \dots, \tag{B19b}$$

$$S_\alpha^{11}(r, r_0) = s'_{11}\xi^2 + s''_{11}\xi^4 + \dots. \tag{B19c}$$

The important conclusion is that the elements of matrix G are series which have either odd or even powers of wavenumber ξ (or frequency ω) with coefficients dependent only on radii and velocities. Elements of the resulting propagator matrix do not contain terms like $\ln\xi$, although these terms are present in functions N_0 and N_1.

Utilizing expansions (B19) and proposition 4 and retaining only main term in the series for each element, we arrive at equation (27) of the main text providing low-frequency representation for the isotropic propagator matrix.

APPENDIX C: GENERALIZED PROPAGATOR MATRIX FOR A FLUID LAYER SANDWICHED BETWEEN TWO SOLID LAYERS

Let us consider a combination of a nonviscous fluid layer sandwiched between two solid layers. In this section we derive a propagator matrix which allows us to connect vectors W taken inside both solids. First, we derive a 2×2 propagator matrix for a fluid layer itself. Second, we build a superstructure 4×4 that takes into account the surrounding solid layers.

For fluid we have $\sigma_{zz} = \sigma_{\phi\phi} = \sigma_{rr}$ and $\sigma_{rz} = 0$. If we consider monochromatic waves (equation (B1)), and use phase velocity $c = \omega/\xi < \infty$, then amplitudes can be found from the equations

$$\rho_f\xi^2c^2u_z - \xi\sigma_{rr} = 0, \tag{C1a}$$

$$\rho_f\xi^2c^2u_r + \partial_r\sigma_{rr} = 0, \tag{C1b}$$

$$\sigma_{rr} = \lambda_f\left[\partial_r u_r + \frac{u_r}{r} + \partial_z u_z\right]. \tag{C1c}$$

The full solution can be described using functions equation (B7) and $\alpha_f^2 = \xi^2[\frac{\rho_f c^2}{\lambda_f} - 1]$ as

$$\begin{bmatrix} \sigma_{rr} \\ u_r \end{bmatrix} = \begin{bmatrix} \rho_f\xi^2c^2J_0(\alpha_f r) & \rho_f\xi^2c^2N_0(\alpha_f r) \\ \alpha_f J_1(\alpha_f r) & \alpha_f N_1(\alpha_f r) \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{C2}$$

where Z_1 and Z_2 are arbitrary constants.

One immediate application of this solution consists in deriving ratio u_r/p for a fluid cylinder in the low frequency limit. In such a case, the solution should satisfy $u_r(0) = 0$ and thus, we have

$$\frac{u_r}{p} = -\frac{u_r}{\sigma_{rr}} = -\frac{\alpha_f J_1(\alpha_f r)}{\rho_f \xi^2 c^2 J_0(\alpha_f r)} \simeq -\frac{\alpha_f^2 r}{2\rho_f \xi^2 c^2}. \tag{C3}$$

If we rewrite this ratio in terms of $X = \lambda_f/(\rho_f c^2)$ then we arrive to equation (8).

Another application of equation (C2) is a propagator matrix for fluid media, which can be derived in the way shown in Appendix B. This matrix relates vectors $V = [\sigma_{rr} \quad r u_r]^t$ at different radii inside the fluid. Using definition (B14) we can write it as

$$\begin{bmatrix} S^{01}_{\alpha_f}(r, r_0) & \rho_f \xi^2 c^2 S^{00}_{\alpha_f}(r_0, r) \\ \frac{1}{\rho_f \xi^2 c^2} S^{11}_{\alpha_f}(r, r_0) & S^{01}_{\alpha_f}(r_0, r) \end{bmatrix}. \tag{C4}$$

Although the two middle components of vector W (they also form vector V) are continuous on all boundaries including solid-fluid interfaces, the remaining two quantities are not. However, we can still construct a generalized matrix $G^{(f)}$, which relates $W(r_{k-1} - 0)$ on the boundary inside the first solid with $W(r_k + 0)$ at the interface inside a second solid. The fluid layer is sandwiched between the solids whereas outer solid layer has fluid outside (or a vacuum if it is the last layer of the composite pipe). The corresponding part of the generalized propagator matrix of the entire system can be expressed as

$$\ldots G^{(f)}_{k+2} G_{k+1}(r_{k+1}, r_k) G^{(f)}_k (r_k, r_{k-1}) G_{k-1} \ldots .$$

Elements of matrix $G^{(f)}$ connecting continuous elements of W coincide with elements of matrix (C4). Tangential stresses σ_{rz} are zero on the boundaries of solid layers with the fluid, thus the fourth row of $G^{(f)}$ is zero. Therefore, we need to obtain the first row. Since there is fluid or vacuum outside the outermost solid layer and hence $\sigma_{rz}(r_{k+1}) = 0$, we can express $u_z(r_k + 0)$ inside the solid through continuous $u_r(r_k)$ and $\sigma_{rr}(r_k)$ by

$$g^{(k+1)}_{41} u_z(r_k + 0) = -g^{(k+1)}_{42} \sigma_{rr}(r_k) - g^{(k+1)}_{43} r_k u_r(r_k),$$

where $g^{(k+1)}_{ij}$ are elements of matrix $G_{k+1}(r_{k+1}, r_k)$. In turn, continuous $u_r(r_k)$ and $\sigma_{rr}(r_k)$ can be expressed via $u_r(r_{k-1})$ and $\sigma_{rr}(r_{k-1})$ by usual propagator matrix (C4) of the fluid layer. As a result, at low frequencies we obtain the following expression

$$G^{(f)}_k(r_k, r_{k-1}) = \begin{bmatrix} 0 & f_{12} & f_{13} & 0 \\ 0 & 1 & \frac{\lambda_f}{X} \xi^2 \ln \frac{r}{r_0} & 0 \\ 0 & \frac{1-X}{\lambda_f} \frac{x_k r_k^2}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C5}$$

where $x_k = 1 - (\frac{r_{k-1}}{r_k})^2$ and

$$f_{12} = -\frac{g^{(k+1)}_{42}}{g^{(k+1)}_{41}} f_{22} - \frac{g^{(k+1)}_{43}}{g^{(k+1)}_{41}} f_{32},$$
$$f_{13} = -\frac{g^{(k+1)}_{42}}{g^{(k+1)}_{41}} f_{23} - \frac{g^{(k+1)}_{43}}{g^{(k+1)}_{41}} f_{33} \tag{C6}$$

are expressed through elements of the middle two rows of $G^{(f)}$.

If we consider a fluid-filled pipe and apply this formalism to describe the innermost fluid layer starting from $r = 0$, then equation (4) can be rewritten as

$$\begin{bmatrix} u_z(r) \\ 0 \\ r u_r(r) \\ 0 \end{bmatrix} G = G(r, r_0) G^{(f)}(r_0, 0) \begin{bmatrix} u_z(0) \\ \sigma_{rr}(0) \\ 0 \\ 0 \end{bmatrix}. \tag{C7}$$

Dispersion equation (6) becomes simply $\tilde{g}_{22} = 0$, where $\tilde{g}_{22}$ is an element of the generalized propagator matrix $GG^{(f)}$. Similarly, we can write dispersion equation for pipe when the pipe consists of n solid layers interchanged with fluid layers. In this case, we should consider $\tilde{g}_{22}$ to be an element of $G_n G^{(f)}_n \ldots G_1 G^{(f)}_1$. Note that the degree of the dispersion equation is $2n$ relative to X which describes n generalized tube waves and n generalized plate waves. Examples of such dispersion equations for $n = 2$ (two concentric pipes filled with fluid) are given by Bakulin *et al.* (2008).