

AN EFFECTIVE MODEL OF A FRACTURED MEDIUM WITH FRACTURES MODELED BY THE SURFACES OF DISCONTINUITY OF DISPLACEMENTS

L. A. Molotkov and A. V. Bakulin

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A medium containing a set of parallel linear slip interfaces is considered. The suggested long wavelength equivalent (effective) model for this medium is shown to be an elastic anisotropic lossless medium. Experimental results demonstrate that this effective model provides an adequate description of wave propagation in media consisting of a stack of parallel rough plates with water-wetted surfaces under a small load. Bibliography: 20 titles.

Presently much importance is attached to the study of wave propagation in fractured media. This is related to both the development of the theory of propagation of seismic waves and to increasing inquiries of applied geophysics. This study is carried out within the framework of several idealizations concerning the microstructure of fractured media [1]. A fruitful idealization of this type is based on the assumption that the structure of the medium under consideration is layered. Within the framework of this idealization, three kinds of low-frequency models, differing from each other in the fracture fillings and in porosity, were formulated. In the first case, fractures of finite sizes are filled with a liquid [2–4]. In the second case, the fillings of fractures and frames are media with moduli of elasticity of the same order; in addition, the fractures may be of arbitrary width [5, 6] or infinitely thin [7].

In the third case, thin fractures are assumed to be filled with an elastic medium, soft in comparison with the frame. As demonstrated in [8], such a medium can be described as a medium with a set of linear slip interfaces. These boundary conditions were introduced in [9]. On the slip interfaces, the displacements have discontinuities but the stresses are continuous. The development of the third kind of fractured models is mainly related to M. Schoenberg and his coauthors [8–12]. The equations of the models in question are of particular interest, because, under certain conditions, they also provide an approximate description of media with compressed oriented spheroidal cracks [11, 13]. The present paper is intended to develop and to theoretically generalize effective models of the third kind, and also to apply them practically.

§ 1. DESCRIPTION OF THE MEDIUM

Consider an elastic medium $0 \leq z \leq H$ with general anisotropy described by the equations of a continuous medium

$$\begin{aligned}\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} &= \rho \frac{\partial^2 u_x}{\partial t^2}, \\ \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} &= \rho \frac{\partial^2 u_y}{\partial t^2}, \\ \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} &= \rho \frac{\partial^2 u_z}{\partial t^2}\end{aligned}\quad (1.1)$$

and the generalized Hooke's law

$$(t_{xx}, t_{yy}, t_{zz}, t_{yz}, t_{xz}, t_{xy})^T = \mathbf{A} \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right)^T, \quad (1.2)$$

where T is the symbol of transposition, $t_{\alpha\beta}$ are the components of the stress tensor, and u_α are the components of the displacement vector ($\alpha, \beta = x, y, z$). The density ρ and the symmetric positive definite

matrix $\mathbf{A}$ characterize the medium. The medium is split by the interfaces $z = h, z = 2h, \dots, z = (n-1)h$, on which the conditions

$$([u_z], [u_y], [u_x])^T = h\tilde{\mathbf{K}}(t_{zz}, t_{yz}, t_{xz})^T, \quad [t_{zz}] = [t_{yz}] = [t_{xz}] = 0, \quad (1.3)$$

are fulfilled; in addition, the quantities $[u_\alpha]$ and $[t_{\alpha\beta}]$ denote the jumps of the displacements u_α and stresses $t_{\alpha\beta}$ on the interfaces $z = ih$ ($i = 1, 2, \dots, n-1$) and are defined by the equalities $[u_\alpha] = u_\alpha(ih+0) - u_\alpha(ih-0)$ and $[t_{\alpha\beta}] = t_{\alpha\beta}(ih+0) - t_{\alpha\beta}(ih-0)$; the matrix $\tilde{\mathbf{K}}$ is symmetric nonnegative definite. In what follows, such a contact will be referred to as the generalized nonrigid contact. Each layer can be considered as inhomogeneous along the z -axis, but, for simplicity of presentation, we shall assume that all layers are homogeneous and have the same parameters. The linear conditions on the outer boundaries $z = 0$ and $z = H$ are inessential. The character and location of a linear source exciting the medium are also inessential.

§ 2. DERIVATION OF THE EFFECTIVE MODEL

We are going to construct an effective model for the medium considered. To this end, in the wave field in this medium we carry out the passage to the limit as $n \rightarrow \infty$, $h \rightarrow 0$, and $H = \text{const}$, as in [4, 6]. Such a passage to the limit can be carried out and fully justified by using the Fourier and Laplace integral transforms [4–6]. However, to shorten the derivation, some physically natural assumptions will be used in this paper.

To carry out this passage to the limit, from Eq. (1.2) we express the derivatives $\frac{\partial u_x}{\partial z}$, $\frac{\partial u_y}{\partial z}$, and $\frac{\partial u_z}{\partial z}$ with respect to z and also the stresses t_{xx} , t_{yy} , and t_{xy} in terms of the displacements u_x, u_y, u_z and stresses t_{xz}, t_{yz} , and t_{zz} . To this end, we represent Eq. (1.2) by the two equalities

$$\begin{aligned} (t_{xx}, t_{yy}, t_{xy})^T &= \mathbf{M} \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)^T + \mathbf{L}^T \left(\frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)^T, \\ (t_{zz}, t_{yz}, t_{xz})^T &= \mathbf{L} \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)^T + \mathbf{N} \left(\frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)^T, \end{aligned} \quad (2.1)$$

where

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} a_{33} & a_{34} & a_{35} \\ a_{34} & a_{44} & a_{45} \\ a_{35} & a_{45} & a_{55} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} a_{31} & a_{32} & a_{36} \\ a_{41} & a_{42} & a_{46} \\ a_{51} & a_{52} & a_{56} \end{pmatrix}. \quad (2.2)$$

Such a splitting of the matrix equation (1.2) into two equations (2.1) was used in [16]. Equations (2.1) imply the relations

$$\begin{aligned} \left(\frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)^T &= \mathbf{N}^{-1}(t_{zz}, t_{yz}, t_{xz})^T - \mathbf{N}^{-1}\mathbf{L} \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)^T, \\ (t_{xx}, t_{yy}, t_{xy})^T &= \mathbf{L}^T \mathbf{N}^{-1}(t_{zz}, t_{yz}, t_{xz})^T + (\mathbf{M} - \mathbf{L}^T \mathbf{N}^{-1} \mathbf{L}) \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)^T, \end{aligned} \quad (2.3)$$

which can be represented by the single matrix equality

$$\left(\frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, t_{xx}, t_{yy}, t_{xy} \right)^T = \mathbf{B} \left(t_{zz}, t_{yz}, t_{xz}, \frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)^T, \quad (2.4)$$

where

$$\mathbf{B} = \frac{1}{A_{345}^{345}} \begin{pmatrix} A_{45}^{45}, & -A_{45}^{35}, & A_{45}^{34}, & -A_{145}^{345}, & -A_{245}^{345}, & -A_{456}^{345} \\ -A_{35}^{45}, & A_{35}^{35}, & -A_{35}^{34}, & A_{135}^{345}, & A_{235}^{345}, & A_{356}^{345} \\ A_{34}^{45}, & -A_{34}^{35}, & A_{34}^{34}, & -A_{134}^{345}, & -A_{234}^{345}, & -A_{346}^{345} \\ A_{345}^{145}, & -A_{345}^{135}, & A_{345}^{134}, & A_{1345}^{1345}, & A_{2345}^{1345}, & -A_{3456}^{1345} \\ A_{345}^{245}, & -A_{345}^{235}, & A_{345}^{234}, & A_{1345}^{2345}, & A_{2345}^{2345}, & -A_{3456}^{2345} \\ A_{345}^{456}, & -A_{345}^{356}, & A_{345}^{346}, & -A_{1345}^{3456}, & -A_{2345}^{3456}, & A_{3456}^{3456} \end{pmatrix}. \quad (2.5)$$

The quantities $A_{\varepsilon\theta\eta\zeta}^{\alpha\beta\gamma\delta}$, $A_{\varepsilon\theta\eta}^{\alpha\beta\gamma}$, and $A_{\varepsilon\theta}^{\alpha\beta}$ are minors of the matrix $\mathbf{A}$, where the superscripts indicate the numbers of the rows included and the subscripts are the numbers of the columns included. Minors of other matrices will be denoted similarly.

The formulas (2.4) and (2.5) just obtained enable us to carry out the passage to the limit in question. We assume that the partial derivatives of the displacements $\tilde{u}_z$, $\tilde{u}_y$, and $\tilde{u}_x$ with respect to z are provided by the relations

$$\frac{\partial \tilde{u}_z}{\partial z} = \frac{\partial u_z}{\partial z} + \frac{[u_z]}{h}, \quad \frac{\partial \tilde{u}_y}{\partial z} = \frac{\partial u_y}{\partial z} + \frac{[u_y]}{h}, \quad \frac{\partial \tilde{u}_x}{\partial x} = \frac{\partial u_x}{\partial z} + \frac{[u_x]}{h}, \quad (2.6)$$

which correspond to the classical definition of the derivative with respect to the decreasing period thickness h . In relations (2.6) we carry out the passage to the limit, after which the first summands of the right-hand sides in (2.6) can be expressed by using the first three scalar equalities from (2.3), while the second summands can be represented by the matrix expression (1.3). The other three equalities from (2.3) remain unchanged after the passage to the limit, but, for definiteness, the limit displacements and stresses are supplied with wavy lines. As a result, matrix relation (2.4) is replaced by the matrix equality

$$\left(\frac{\partial \tilde{u}_z}{\partial z}, \frac{\partial \tilde{u}_y}{\partial z} + \frac{\partial \tilde{u}_z}{\partial y}, \frac{\partial \tilde{u}_x}{\partial z} + \frac{\partial \tilde{u}_z}{\partial x}, \tilde{t}_{xx}, \tilde{t}_{yy}, \tilde{t}_{xy} \right)^T = \mathbf{C} \left(\tilde{t}_{zz}, \tilde{t}_{yz}, \tilde{t}_{xz}, \frac{\partial \tilde{u}_x}{\partial x}, \frac{\partial \tilde{u}_y}{\partial y}, \frac{\partial \tilde{u}_x}{\partial y} + \frac{\partial \tilde{u}_y}{\partial x} \right)^T, \quad (2.7)$$

where

$$\mathbf{C} = \mathbf{B} + \mathbf{K}, \quad (2.8)$$

$$\mathbf{K} = \begin{pmatrix} k_{11}, & k_{12}, & k_{13}, & 0, & 0, & 0 \\ k_{12}, & k_{22}, & k_{23}, & 0, & 0, & 0 \\ k_{13}, & k_{23}, & k_{33}, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}, \quad (2.9)$$

and k_{ij} are the entries of the matrix $\tilde{\mathbf{K}}$. Further transformations consist of deriving the relation

$$(\tilde{t}_{xx}, \tilde{t}_{yy}, \tilde{t}_{zz}, \tilde{t}_{yz}, \tilde{t}_{xz}, \tilde{t}_{xy})^T = \mathbf{D} \left(\frac{\partial \tilde{u}_x}{\partial x}, \frac{\partial \tilde{u}_y}{\partial y}, \frac{\partial \tilde{u}_z}{\partial z}, \frac{\partial \tilde{u}_y}{\partial z} + \frac{\partial \tilde{u}_z}{\partial y}, \frac{\partial \tilde{u}_x}{\partial z} + \frac{\partial \tilde{u}_z}{\partial x}, \frac{\partial \tilde{u}_x}{\partial y} + \frac{\partial \tilde{u}_y}{\partial x} \right)^T \quad (2.10)$$

from the matrix equality (2.7), where

$$\mathbf{D} = \frac{1}{C_{123}^{123}} \begin{pmatrix} C_{1234}^{1234} & C_{1235}^{1234} & C_{123}^{234} & -C_{123}^{134} & C_{123}^{124} & C_{1236}^{1234} \\ C_{1234}^{1235} & C_{1235}^{1235} & C_{123}^{235} & -C_{123}^{135} & C_{123}^{125} & C_{1236}^{1235} \\ -C_{234}^{123} & C_{235}^{123} & C_{23}^{23} & -C_{23}^{13} & C_{23}^{12} & -C_{236}^{123} \\ C_{134}^{123} & C_{135}^{123} & -C_{13}^{23} & C_{13}^{13} & -C_{13}^{12} & C_{136}^{123} \\ -C_{124}^{123} & -C_{125}^{123} & C_{12}^{23} & -C_{12}^{13} & C_{12}^{12} & -C_{126}^{123} \\ C_{1234}^{1236} & C_{1235}^{1236} & C_{123}^{236} & -C_{123}^{136} & C_{123}^{126} & C_{1236}^{1236} \end{pmatrix}. \quad (2.11)$$

Matrix equation (2.10) is Hooke's law for the effective model. To ascertain that the effective model is an elastic medium, it is necessary to prove that the matrix $\mathbf{D}$ is symmetric and positive definite and to show that the equations of a continuous medium for the effective model correspond to an elastic medium.

If we use the equalities

$$A_{\varepsilon\theta\eta\zeta}^{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta}^{\varepsilon\theta\eta\zeta}, \quad A_{\varepsilon\theta\eta}^{\alpha\beta\gamma} = A_{\alpha\beta\gamma}^{\varepsilon\theta\eta}, \quad A_{\varepsilon\theta}^{\alpha\beta} = A_{\alpha\beta}^{\varepsilon\theta}, \quad (2.12)$$

valid for minors of the symmetric matrix $\mathbf{A}$, then we arrive at the conclusion that the entries of the matrix $\mathbf{B}$ located symmetrically with respect to the principal diagonal are either equal,

$$b_{21} = b_{12}, \quad b_{31} = b_{13}, \quad b_{32} = b_{23}, \quad b_{54} = b_{45}, \quad b_{64} = b_{46}, \quad b_{65} = b_{56}, \quad (2.13)$$

or have opposite signs

$$\begin{aligned} b_{41} &= -b_{14}, \quad b_{51} = -b_{15}, \quad b_{61} = -b_{16}, \quad b_{42} = -b_{24}, \quad b_{52} = -b_{25}, \\ b_{62} &= -b_{26}, \quad b_{43} = -b_{34}, \quad b_{53} = -b_{35}, \quad b_{63} = -b_{36}. \end{aligned} \quad (2.14)$$

In view of the symmetry of the matrix $\mathbf{K}$, relations analogous to (2.13) and (2.14) are also valid for the entries c_{ij} of the matrix $\mathbf{C}$. From equalities (2.13) written for the entries of the matrix $\mathbf{C}$ it follows that the matrix

$$\begin{pmatrix} C_{23}^{23} & -C_{23}^{13} & C_{23}^{12} \\ -C_{13}^{23} & C_{13}^{13} & -C_{13}^{12} \\ C_{12}^{23} & -C_{12}^{13} & C_{12}^{12} \end{pmatrix} \quad (2.15)$$

is symmetric. Taking into account relations (2.14) insures that the matrix

$$\begin{pmatrix} C_{1234}^{1234} & C_{1235}^{1234} & C_{1236}^{1234} \\ C_{1234}^{1235} & C_{1235}^{1235} & C_{1236}^{1235} \\ C_{1234}^{1236} & C_{1235}^{1236} & C_{1236}^{1236} \end{pmatrix} \quad (2.16)$$

also proves to be symmetric. Therefore, the entries of the matrix $\mathbf{D}$ located symmetrically with respect to the principal diagonal are equal.

To establish the positive definiteness of $\mathbf{D}$, it is sufficient to prove the inequalities

$$d_{66} > 0, \quad D_{56}^{56} > 0, \quad D_{456}^{456} > 0, \quad D_{3456}^{3456} > 0, \quad D_{23456}^{23456} > 0, \quad \det \mathbf{D} > 0. \quad (2.17)$$

By using the properties of determinants, for the left-hand sides of inequalities (2.17) we obtain the expressions

$$d_{66} = C_{12}^{12}, \quad D_{56}^{56} = \begin{vmatrix} C_{13}^{13} & C_{13}^{12} \\ C_{12}^{13} & C_{12}^{12} \end{vmatrix}, \quad D_{456}^{456} = \begin{vmatrix} C_{23}^{23} & C_{23}^{13} & C_{23}^{12} \\ C_{13}^{23} & C_{13}^{13} & C_{13}^{12} \\ C_{12}^{23} & C_{12}^{13} & C_{12}^{12} \end{vmatrix}, \quad (2.18)$$

$$D_{3456}^{3456} = c_{66} \cdot C_{123}^{123} D_{456}^{456}, \quad D_{23456}^{23456} = C_{123}^{123} C_{56}^{56} D_{456}^{456}, \quad \det \mathbf{D} = C_{123}^{123} C_{456}^{456} D_{456}^{456}.$$

To show that all the minors in (2.18) are positive, we use the following theorem. If a matrix $\mathbf{H}$ is positive definite, then every matrix $\tilde{\mathbf{H}}$ whose entries are minors of the matrix $\mathbf{H}$ of the same order is also positive definite. The proof of this theorem, based on applying Gantmacher's theorem [14], can be found in [15].

Since the original matrix $\mathbf{A}$ is positive definite, all its principal minors are positive. Thus, in view of the above theorem, the matrices

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}, \quad \begin{pmatrix} b_{44} & b_{45} & b_{46} \\ b_{45} & b_{55} & b_{56} \\ b_{46} & b_{56} & b_{66} \end{pmatrix} \quad (2.19)$$

are positive definite. Since the sum of a positive definite and a nonnegative definite matrix is positive definite, the matrices

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \quad \begin{pmatrix} c_{44} & c_{45} & c_{46} \\ c_{45} & c_{55} & c_{56} \\ c_{46} & c_{56} & c_{66} \end{pmatrix} \quad (2.20)$$

are also positive definite. Applying the same theorem once again, we ascertain that the matrix (2.15) is positive definite. Finally, from the positive definiteness of the matrices (2.15) and (2.20) it follows that all the quantities in (2.18) are positive; therefore, the matrix $\mathbf{D}$ is positive definite.

Since the equations of a continuous medium (1.1) are independent of the type of anisotropy and since they are not influenced by boundary conditions (1.3) when the passage to the limit is carried out, after this passage they keep the same form

$$\begin{aligned} \frac{\partial \tilde{t}_{xx}}{\partial x} + \frac{\partial \tilde{t}_{xy}}{\partial y} + \frac{\partial \tilde{t}_{xz}}{\partial z} &= \rho \frac{\partial^2 \tilde{u}_x}{\partial t^2}, \\ \frac{\partial \tilde{t}_{xy}}{\partial x} + \frac{\partial \tilde{t}_{yy}}{\partial y} + \frac{\partial \tilde{t}_{yz}}{\partial z} &= \rho \frac{\partial^2 \tilde{u}_y}{\partial t^2}, \\ \frac{\partial \tilde{t}_{xz}}{\partial x} + \frac{\partial \tilde{t}_{yz}}{\partial y} + \frac{\partial \tilde{t}_{zz}}{\partial z} &= \rho \frac{\partial^2 \tilde{u}_z}{\partial t^2} \end{aligned} \quad (2.21)$$

but under the new notation. The form of Eqs. (2.21) and the positive definiteness and symmetry of the matrix $\mathbf{D}$ imply the following main conclusion of Sec. 2. The effective model of the medium from Sec. 1 described by Eqs. (2.10) and (2.11) is an elastic anisotropic medium for an arbitrary type of anisotropy of the original medium and for an arbitrary generalized nonrigid contact (1.3) with a nonnegative definite matrix $\tilde{\mathbf{K}}$.

To conclude this section, we point out that, in the case where the period is a layer inhomogeneous along the z -axis with welded contacts on inner interfaces, the matrix $\mathbf{B}$ and the density ρ should be averaged on the period and replaced in equalities (2.8) and (2.21) by $\bar{\mathbf{B}}$ and $\bar{\rho}$, respectively. Since the required properties of the matrix $\mathbf{B}$ are preserved in the averaged matrix $\bar{\mathbf{B}}$, in this case too, the effective model is an elastic anisotropic medium.

A layered anisotropic elastic periodic medium with welded contacts on all interfaces is the particular case of an inhomogeneous periodic elastic medium with contacts (1.3) on the inner interfaces between the periods. This case corresponds to the zero matrices $\mathbf{K}$ and $\tilde{\mathbf{K}}$. For the effective model of such a medium relation (2.8) is replaced by the equality

$$\mathbf{C} = \bar{\mathbf{B}}, \quad (2.22)$$

whereas the density ρ in Eqs. (2.21) is substituted for by the averaged density $\bar{\rho}$. This effective model was constructed in [16], but there the positive definiteness of the matrix $\mathbf{D}$ was not proved. This property can be established using the approach developed in this section.

§ 3. ENERGY INVESTIGATIONS

Consider a homogeneous elastic medium $0 \leq z \leq H$ with one inner interface $z = H/2$ on which the conditions (1.3) of generalized nonrigid contact are fulfilled; h is a parameter with length dimension. At every point of this medium the differential law of conservation of energy

$$\frac{\partial W_k}{\partial t} + \frac{\partial W_p}{\partial t} = \frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z} \quad (3.1)$$

is obeyed, where W_k and W_p are the densities of the kinetic and potential energies, and p_x, p_y , and p_z are the fluxes of energy along the x -, y -, and z -axes, respectively. We integrate formula (3.1) over the entire volume V of the given medium and assume that the conditions of the energy closedness of the medium

$$p_z|_{z=\pm H} = 0, \quad p_x|_{x=\pm\infty} = 0, \quad p_y|_{y=\pm\infty} = 0 \quad (3.2)$$

are fulfilled. Taking into account conditions (3.2) and the singularities on the interface $z = H/2$, which is denoted by S , we express the result of integration by the relation

$$\frac{\partial}{\partial t} \iiint_V (W_k + W_p) dV + \iint_S [P_z] ds = 0. \quad (3.3)$$

The flux p_z along the z -axis and its jump on the interface $z = H/2$ are represented by the equalities

$$\begin{aligned} p_z &= t_{zz} \frac{\partial u_z}{\partial t} + t_{yz} \frac{\partial u_y}{\partial t} + t_{xz} \frac{\partial u_x}{\partial t}, \\ [p_z] &= t_{zz} \frac{\partial [u_z]}{\partial t} + t_{yz} \frac{\partial [u_y]}{\partial t} + t_{xz} \frac{\partial [u_x]}{\partial t}. \end{aligned} \quad (3.4)$$

Taking into account conditions (1.3), we express the jump $[p_z]$ of the energy flux by the formula

$$[p_z] = h \frac{\partial F}{\partial t}, \quad (3.5)$$

where

$$F = \frac{1}{2} k_{11} t_{zz}^2 + k_{12} t_{zz} t_{yz} + k_{13} t_{zz} t_{xz} + \frac{1}{2} k_{22} t_{yz}^2 + k_{23} t_{xz} t_{yz} + \frac{1}{2} k_{33} t_{xz}^2. \quad (3.6)$$

Upon substituting expression (3.5) into equality (3.3), we have the relation

$$\iiint_V (W_k + W_p) dV + h \iint_S F ds = \text{const}, \quad (3.7)$$

where the right-hand side is independent of time. If we consider $n - 1$ interfaces $z = z_i = iH/n = ih$ ($i = 1, 2, \dots, n - 1$) rather than one and retain boundary conditions (1.3) and (3.2), then equality (3.7) is replaced by the relation

$$\iiint_V (W_k + W_p) dV + h \sum_{i=1}^{n-1} \iint_S F(z_i) ds = \text{const}. \quad (3.8)$$

Upon passing to the limit as $n \rightarrow \infty$, $h \rightarrow 0$, and $H = \text{const}$, the sum $h \sum_{i=1}^{n-1} F(z_i)$, as a Riemann sum, turns

into the definite integral $\int_0^H F(z) dz$, and the equality (3.8) takes the form

$$\iiint_V (W_k + W_p + F) dV = \text{const}. \quad (3.9)$$

Equality (3.9) can be considered as the law of conservation of energy; in this case, the potential energy is expressed as a sum of two terms. The first term W_p is related to the density of the elastic potential energy and is expressed via the known relation

$$2W_p = \frac{\partial u_x}{\partial x} t_{xx} + \frac{\partial u_y}{\partial y} t_{yy} + \frac{\partial u_z}{\partial z} t_{zz} + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) t_{yz} + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) t_{xz} + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) t_{xy}. \quad (3.10)$$

The second term F represents the density of the potential energy of distributed fractures and is expressed via (3.6). According to Eqs. (3.7) and (3.8), in the case of one or several linear slip interfaces, an additional potential energy also arises, but it is concentrated on the interfaces instead of being volume-distributed.

Thus, the effective model is not dissipative, and on the boundaries with generalized nonrigid contacts a special potential energy with surface density arises. After the passage to the limit, when the number of such boundaries becomes infinite and they are uniformly distributed in the medium considered, this kind of potential energy converts into the volumetric one.

The density of the kinetic energy W_k of the model, as in the conventional elastic media, is given by

$$W_k = \frac{\rho}{2} \left[\left(\frac{\partial u_x}{\partial t} \right)^2 + \left(\frac{\partial u_y}{\partial t} \right)^2 + \left(\frac{\partial u_z}{\partial t} \right)^2 \right]. \quad (3.11)$$

If we use expressions (3.6), (3.10), and (3.11) for the densities of the potential and kinetic energies and Hooke's law (1.2), then on the basis of Lagrange's equations we can derive Eqs. (2.10) and (2.11) of the effective model. Comparison of the two approaches to deriving these equations also provides another representation of the matrix $\mathbf{D}$ in the form

$$\mathbf{D}^{-1} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} & \tilde{a}_{15} & \tilde{a}_{16} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{23} & \tilde{a}_{24} & \tilde{a}_{25} & \tilde{a}_{26} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} + k_{11} & \tilde{a}_{34} + k_{12} & \tilde{a}_{35} + k_{13} & \tilde{a}_{36} \\ \tilde{a}_{14} & \tilde{a}_{24} & \tilde{a}_{34} + k_{12} & \tilde{a}_{44} + k_{22} & \tilde{a}_{45} + k_{23} & \tilde{a}_{46} \\ \tilde{a}_{15} & \tilde{a}_{25} & \tilde{a}_{35} + k_{13} & \tilde{a}_{45} + k_{23} & \tilde{a}_{55} + k_{33} & \tilde{a}_{56} \\ \tilde{a}_{16} & \tilde{a}_{26} & \tilde{a}_{36} & \tilde{a}_{46} & \tilde{a}_{56} & \tilde{a}_{66} \end{pmatrix}, \quad (3.12)$$

where $\tilde{a}_{ij}$ are the entries of the inverse matrix $\mathbf{A}^{-1}$.

Finally, we note that the conclusion about the absence of absorption in the effective model of the medium considered can also be deduced from the results of [17], where a more general type of contact was examined but only the tangential displacements were allowed to be discontinuous on the interfaces.

§ 4. INVESTIGATION OF A MEDIUM WITH THIN SOFT LAYERS

Our purpose is to clarify in what elastic systems boundary conditions (1.3) are approximately satisfied. To this end, we preliminarily derive a special expression for the density of the potential energy in an elastic medium with general anisotropy. As is known, the density of the potential energy is given by the relation

$$W_p = \frac{1}{2} \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) (t_{xx}, t_{yy}, t_{xy})^T + \frac{1}{2} \left(\frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) (t_{zz}, t_{yz}, t_{xz})^T. \quad (4.1)$$

Into Eq. (4.1) we insert the second formula from (2.3) and also the relation

$$\left(\frac{\partial u_z}{\partial z}, \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = (t_{zz}, t_{yz}, t_{xz}) N^{-1} - \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_x}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) L^T N^{-1}, \quad (4.2)$$

which is obtained by transposing the first equation from (2.3). As a result, we arrive at the required formula

$$\begin{aligned} W_p &= \frac{1}{2} \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) (\mathbf{M} - \mathbf{L}^T \mathbf{N}^{-1} \mathbf{L}) \times \\ &\times \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)^T + \frac{1}{2} (t_{zz}, t_{yz}, t_{xz}) \mathbf{N}^{-1} (t_{zz}, t_{yz}, t_{xz})^T \end{aligned} \quad (4.3)$$

for the density of the potential energy.

Consider a layered periodic elastic medium $0 \leq z \leq H$ with n periods, where each period consists of two homogeneous layers of thicknesses h_1 and h_2 . Layers may possess general anisotropy and are assumed to be in welded contact. We also assume that the second layers are thin and soft as compared with the first ones, whereas the coefficients $a_{ij}^{(1)}$ and $a_{ij}^{(2)}$ of Hooke's law and the thicknesses satisfy the conditions

$$h_2 \ll h_1, \quad a_{ij}^{(2)} \ll a_{ij}^{(1)}; \quad (4.4)$$

in addition, the ratios $a_{ij}^{(1)}/h_1$ and $a_{ij}^{(2)}/h_2$ are assumed to be of the same order of magnitude.

If we take into account the continuity of the stresses and derivatives of the displacements involved in (4.3) on the interfaces, the relatively small thickness of the second layer, and assumptions (4.4), then we conclude that the first term in expression (4.3) for the density $W_p^{(2)}$ of the potential energy in the second layer is small in comparison with the second term. Thus, the value of $W_p^{(2)}$ is approximately given by

$$W_p^{(2)} = \frac{1}{2} \left(t_{zz}^{(2)}, t_{yz}^{(2)}, t_{xz}^{(2)} \right) \mathbf{N}^{(2)-1} \left(t_{zz}^{(2)}, t_{yz}^{(2)}, t_{xz}^{(2)} \right)^T, \quad (4.5)$$

where the superscripts indicate that the stresses and the matrix $\mathbf{N}$ refer to the second layer.

In the medium considered, we carry out the passage to the limit as $n \rightarrow \infty$, $h_1 \rightarrow 0$, $h_2 \rightarrow 0$, $h_1/h_2 = \text{const}$, and $H = n(h_1 + h_2) = \text{const}$. Similar passages were repeatedly applied in deriving equations of effective models and, in particular, in [16], where an anisotropic elastic periodic medium with welded contacts on the interfaces was investigated. After the passage to the limit, by taking into account the first inequality from (4.4), for the density of the potential energy we obtain

$$W_p = W_p^{(1)} + \frac{h_2}{h_1} W_p^{(2)}. \quad (4.6)$$

Relations (4.5) and (4.6) together with Eqs. (3.9) and (3.6) show that the densities of the potential energy in the medium with thin soft layers and in the effective model are represented by similar expressions. Now we determine under what conditions the layered medium under consideration has the same properties as the effective model. Comparison of the two expressions for the densities of the potential energy leads to the equality

$$\tilde{\mathbf{K}} = h_2 \mathbf{N}^{(2)-1} / h_1 \quad (4.7)$$

for the matrix $\tilde{\mathbf{K}}$ that specifies the generalized nonrigid contacts on the inner interfaces of the medium described in Sec. 1. Before the passage to the limit, in the layered medium to these interfaces there correspond artificial boundaries replacing soft thin layers. In both media, an additional potential energy given by (3.6) and (4.5) is associated with the interfaces. This energy causes a jump of the energy flux on these artificial boundaries, which is provided by the formula

$$[p_z] = \frac{h_2}{2} \frac{\partial}{\partial t} \left[(t_{zz}^{(2)}, t_{yz}^{(2)}, t_{xz}^{(2)}) \mathbf{N}^{(2)-1} (t_{zz}^{(2)}, t_{yz}^{(2)}, t_{xz}^{(2)})^T \right]. \quad (4.8)$$

A similar expression for the jump of the energy flux on the interface of the medium introduced in Sec. 1 is provided by (3.4)–(3.6).

With the jump of the energy flux the discontinuities of stresses and displacements on the artificial interfaces are associated. Since the coefficients $a_{ij}^{(2)}$ of Hooke's law are small, the stresses $t_{zz}^{(2)}, t_{zy}^{(2)}, t_{zx}^{(2)}$ vary along the z -axis in each second layer much slower than the displacements $u_x^{(2)}, u_y^{(2)},$ and $u_z^{(2)}$. Therefore, in the case where the thickness h_2 is sufficiently small, the stresses $t_{zz}^{(2)}, t_{zy}^{(2)},$ and $t_{zx}^{(2)}$ may be considered as equal on both interfaces of each second layer. At the same time, in order that the displacements $u_x^{(2)}, u_y^{(2)}, u_z^{(2)}$ take different values on the above interfaces and have jumps on the artificial boundaries, which are described by conditions (1.3) and the matrix (4.7), the value of h_2 should not be too small. Thus, the conditions on the interfaces of soft thin layers may indeed lead to the boundary conditions of generalized nonrigid contacts.

If, in addition to inequality (4.4), h_2 satisfies the condition

$$h_1/a_{ij}^{(1)} \gg h_2/a_{ij}^{(2)}, \quad (4.9)$$

then the variations of the displacements $u_x^{(2)}, u_y^{(2)}, u_z^{(2)}$ along the z -axis within second layers may be ignored. In this case, the matrix $\tilde{\mathbf{K}}$ defined by (4.7) is zero and the contacts on the artificial interfaces prove to be welded; furthermore, because of their small thicknesses, second layers are actually absent.

Now we turn our attention to the matrix $\tilde{\mathbf{K}}$ given by relation (4.7) and specifying the discontinuities of displacements on the artificial interfaces. The symmetry and positive definiteness of the matrix $\mathbf{N}^{(2)}$ for the second layer imply that $\tilde{\mathbf{K}}$ is also symmetric and positive definite. In the cases where the coefficients $a_{ij}^{(2)}$ of Hooke's law differ from each other in such a way that some of them may be ignored in comparison with others, the matrix $\tilde{\mathbf{K}}$ can turn out to be positive semidefinite.

If the fracture-filling material possesses general anisotropy, then the contact (1.3) between media is characterized by 6 parameters; however, as was shown in [18], the number of independent parameters can be reduced to 5. In the case of a monoclinic filling medium, the matrix $\tilde{\mathbf{K}}$ is determined by 4 independent parameters. In other cases where one of the principal anisotropy axis is normal to the interfaces, the matrix $\tilde{\mathbf{K}}$ is diagonal. If the filling medium is orthotropic, then all 3 diagonal entries of the matrix $\tilde{\mathbf{K}}$ are distinct. In the case of the tetragonal, trigonal, hexagonal, or cubic symmetry, as well as for an isotropic filling material, the matrix $\tilde{\mathbf{K}}$ is of the form

$$\tilde{\mathbf{K}} = \begin{pmatrix} K_N & 0 & 0 \\ 0 & K_T & 0 \\ 0 & 0 & K_T \end{pmatrix}, \quad (4.10)$$

where $K_N > 0$ and $K_T > 0$. Media with interfaces described by the matrix (4.10) are the most important among periodic elastic media with contacts (1.3). These media were investigated in [10].

§ 5. INVESTIGATION OF EFFECTIVE MODELS

Let the medium defined in Sec. 1 be isotropic and let boundary conditions (1.3) be supplemented with relation (4.10). In accordance with Sec. 2, in this case the effective model is characterized by the quasidiagonal matrix

$$\mathbf{D} = \left[\mathbf{D}_1, \frac{\mu}{1 + K_T\mu}, \frac{\mu}{1 + K_T\mu}, \mu \right], \quad (5.1)$$

where

$$\mathbf{D}_1 = \frac{1}{1 + (\lambda + 2\mu)K_N} \begin{pmatrix} \lambda + 2\mu + 4\mu(\lambda + \mu)K_N, & \lambda(1 + 2\mu K_N), & \lambda \\ \lambda(1 + 2\mu K_N), & \lambda + 2\mu + 4\mu(\lambda + \mu)K_N, & \lambda \\ \lambda, & \lambda, & \lambda + 2\mu \end{pmatrix}. \quad (5.2)$$

From the structure of matrices (5.1) and (5.2) entering in Hooke's law and from Eqs. (2.21) it follows that the effective model is transversally-isotropic with the isotropy planes $z = \text{const}$. This conclusion conforms with the results of [10]. If we introduce two dimensionless parameters δ_N and δ_T via

$$K_N = \frac{\delta_N}{(\lambda + 2\mu)(1 - \delta_N)}, \quad K_T = \frac{\delta_T}{\mu(1 - \delta_T)}, \quad (5.3)$$

then for the matrix $\mathbf{D}$ we derive another representation

$$\mathbf{D} = [\mathbf{D}_1, \mu(1 - \delta_T), \mu(1 - \delta_T), \mu],$$

$$\mathbf{D}_1 = \begin{pmatrix} (\lambda + 2\mu) \left[1 - \frac{\lambda^2 \delta_N}{(\lambda + 2\mu)^2} \right], & \lambda \left(1 - \frac{\lambda \delta_N}{\lambda + 2\mu} \right), & \lambda(1 - \delta_N) \\ \lambda \left(1 - \frac{\lambda \delta_N}{\lambda + 2\mu} \right), & (\lambda + 2\mu) \left[1 - \frac{\lambda^2 \delta_N}{(\lambda + 2\mu)^2} \right], & \lambda(1 - \delta_N) \\ \lambda(1 - \delta_N), & \lambda(1 - \delta_N), & (\lambda + 2\mu)(1 - \delta_N) \end{pmatrix}, \quad (5.4)$$

which completely agrees with the results of [10].

Consider the limiting cases of media with Hooke's law given by Eqs. (5.1) and (5.2), or by (5.4). To this end, from relations (5.3) we obtain the expressions

$$\delta_N = \frac{(\lambda + 2\mu)K_N}{1 + (\lambda + 2\mu)K_N}, \quad \delta_T = \frac{\mu K_T}{1 + \mu K_T}. \quad (5.5)$$

As it follows from (5.5), the dimensionless parameters δ_N and δ_T vary in the interval $[0, 1]$. The case $\delta_N = 0$ ($\delta_T = 0$) corresponds to the absence of discontinuities of the normal (tangential) displacements on the interfaces. If $\delta_N = \delta_T = 0$, then the contact is welded and the effective model is isotropic. If $\delta_T = 1$ and $\delta_N = 0$, then the contact on the interfaces of the medium considered is the slipping one. For this case, the effective model was constructed and investigated in [5, 7]. In this model, SV waves cannot propagate while SH waves propagate only along the planes $z = \text{const}$ because the slipping interfaces serve as barriers for the propagation of the SH waves. In the case $\delta_N = 1$ and $\delta_T = 0$, the matrix $\mathbf{D}$ is represented in the form

$$\mathbf{D} = \left[\frac{1}{\lambda + 2\mu} \begin{pmatrix} 4\mu(\lambda + \mu), & 2\lambda\mu \\ 2\lambda\mu, & 4\mu(\lambda + \mu) \end{pmatrix}, 0, \mu, \mu, \mu \right]. \quad (5.6)$$

The specific property of this model consists in that the quasilongitudinal wave can propagate in it only along the planes $z = \text{const}$. For this wave, the medium turns out to be fully separated by the interfaces. The velocity of the quasilongitudinal wave coincides with the velocity

$$2\sqrt{\frac{\mu(\lambda + \mu)}{\rho(\lambda + 2\mu)}} \quad (5.7)$$

of the plate wave propagating in a free elastic layer.

§ 6. INVESTIGATION OF MEDIA WITH ROUGH INTERFACES AND FRACTURES

The effective model derived in Sec. 2 is applicable not only to thin soft layers but also to rocks containing natural fractures and to media consisting of plates with rough surfaces under an exterior load. Reflection and refraction of waves on the interface represented by a single natural fracture were investigated in [19], where it was demonstrated that such processes are adequately described if we assume that the contact on this interface is the generalized nonrigid one.

To demonstrate the applicability of the effective model in question to describe wave propagation in media with rough interfaces, we make use of the results of a laboratory experiment presented in [20]. In this experiment, the velocities of shear waves in a stack of plexiglas plates with rough surfaces wetted by water were measured. To this stack a small constant load normal to the fractures was applied. In this

experiment, the intensity n of fractures, i.e., the number of fractures per unit length equal to 1 in. = 2.54 cm, was varied in the direction perpendicular to the fracture planes. This variation was effected by using plates of different thicknesses (from 0.08 to 2.54 cm or from 1/32 to 1 in.). A total of 14 models were used; each of them was of the summarized thickness equal to 4 in. = 10.16 cm. The dominant amplitude in the signal spectrum was about 50 kHz, and the corresponding wavelength of the shear wave in plexiglas was 2.74 cm.

The velocities of shear waves were measured along fractures, and two types of wave polarization were considered: the polarization parallel to the fracture planes and that perpendicular to them. Measurements resulted in the two velocities $v_{\parallel}$ and $v_{\perp}$ considered as functions of the fracture intensity n . The velocity $v_{\parallel}$ proved to be nearly constant for all models, and its value was close to the value of the shear wave velocity in a solid plexiglas block. Figure 1 exhibits the experimentally established dependence of the ratio $v_{\parallel}/v_{\perp}$ upon the fracture intensity n . This dependence is represented by the straight line passing through the point (0, 1).

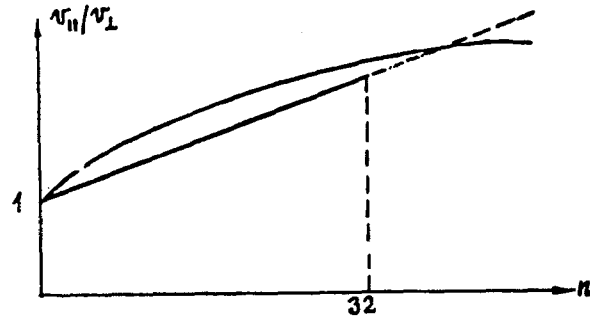


FIG. 1

To explain this experimental behavior, we apply the results of the above investigations of the effective model to the case where the host medium is isotropic and the matrix $\tilde{\mathbf{K}}$ is specified by (4.10). In accordance with (5.1), the velocities of the shear waves are determined by the relations

$$v_{\parallel} = \sqrt{\frac{\mu}{\rho}}, \quad v_{\perp} = \sqrt{\frac{\mu}{\rho(1 + K_T\mu)}}. \quad (6.1)$$

From the first formula in (6.1) it follows that the velocity $v_{\parallel}$ is independent of the boundary discontinuities and is equal to the velocity of the shear waves in a massif, which conforms to the experimental data. To explain the graphs in Fig. 1, we write the relation

$$v_{\parallel}/v_{\perp} = \sqrt{1 + K_T\mu}, \quad (6.2)$$

based on formulas (6.1). The value of K_T is defined as the jump of the tangential displacement under the unit shear stress when the fracture spacing h is equal to the unit length, i.e., to 1 in. If $h \neq 1$ in., then, since K_T is the same for all fractures in all the models, by summing the values of K_T corresponding to the interval of unit length, we obtain the expression

$$v_{\parallel}/v_{\perp} = \sqrt{1 + qn}, \quad (6.3)$$

where $q = \mu K_T$. For

$$n \ll 1/q \quad (6.4)$$

we have the approximate relation

$$v_{\parallel}/v_{\perp} = 1 + \frac{q}{2} n, \quad (6.5)$$

which reveals the linear dependence detected in the experiment. The straight line best fitted to the experimental points is defined by the parameter $q = 2.572 \cdot 10^{-2}$ in. In Fig. 1, through the same points we

drew the parabola (6.3) with $q = 3.301 \cdot 10^{-2}$ in. The discrepancy between the straight line and parabola for $0 \leq n \leq 32$ lies within the range of experimental errors. Thus, for $0 \leq n \leq 32$ the experimental and theoretical results are in good agreement, which confirms the applicability of the model used.

Similar laboratory experiments with rough plexiglas plates of thickness 0.7 mm were described in [10]. In that paper, the possibility of using the effective model to describe wave propagation in a stack of rough plates was confirmed for the case where the wavelength exceeds the period thickness by more than 2.5 times.

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Literature Cited

- 1 J. A. Hudson, "Overall properties of heterogeneous materials," *Geophys. J. Int.*, **107**, 505-511 (1991).
- 2 L. A. Molotkov, "The equivalence of periodically stratified and transversally isotropic media," *Zap. Nauchn. Semin. LOMI*, **89**, 219-233 (1979).
- 3 L. A. Molotkov, *The Matrix Method in the Theory of Wave Propagation in Layered Elastic and Fluid Media* [in Russian], Leningrad (1984).
- 4 L. A. Molotkov, "On a new approach to deriving the equations of an averaged effective model of periodic media," *Zap. Nauchn. Semin. LOMI*, **195** 82-102 (1991).
- 5 L. A. Molotkov and A. E. Khilo, "Investigation of single- and multiphase effective models describing periodic media," *Zap. Nauchn. Semin. LOMI*, **140** 105-122 (1984).
- 6 L. A. Molotkov, "On an effective model describing a layered periodic elastic medium with slide contacts on the interfaces," *Zap. Nauchn. Semin. POMI*, **210**, 192-212 (1994).
- 7 L. A. Molotkov, "On peculiarities of wave propagation in layered models of fractured media," *Zap. Nauchn. Semin. LOMI*, **173**, 123-133 (1988).
- 8 M. Schoenberg, "Reflection of elastic waves from periodically stratified media with interfacial slip," *Geophys. Prospecting*, **31**, 265-292 (1983).
- 9 M. Schoenberg, "Elastic wave behavior across linear slip interfaces," *J. Acoust. Soc. Am.*, **68**, No. 5, 1516-1521 (1980).
- 10 C.-J. Hsu and M. Schoenberg, "Elastic waves through a simulated fractured medium," *Geophysics*, **58**, No. 7, 964-977 (1993).
- 11 M. Schoenberg and J. Douma, "Elastic wave propagation in media with parallel fractures and aligned cracks," *Geophys. Prospecting*, **36**, 571-590 (1988).
- 12 M. Schoenberg and F. Muir, "A calculus for finely layered anisotropic media," *Geophysics*, **54**, No. 5, 581-589 (1989).
- 13 J. A. Hudson, "Wave speeds and attenuation of elastic waves in material containing cracks," *Geophys. J. Royal Astr. Soc.*, **64**, 133-150 (1981).
- 14 F. R. Gantmacher, *Theory of Matrices* [in Russian], Moscow (1967).
- 15 L. A. Molotkov, "An effective model of an elastic anisotropic medium with fluid-filled fractures," in: *Problems in the Dynamic Theory of Propagation of Seismic Waves* [in Russian], Vol. 29 (1989), pp. 14-29.
- 16 L. A. Molotkov and A. E. Khilo, "Effective media for periodic anisotropic systems," *Zap. Nauchn. Semin. LOMI*, **128**, 130-138 (1983).
- 17 L. A. Molotkov and A. E. Khilo, "Effective models of layered media with linear contacts of the general type," *Zap. Nauchn. Semin. LOMI*, **156**, 148-157 (1986).
- 18 E. Berg, J. A. Hood, and G. J. Fryer, "Reduction of the general fracture compliance matrix **Z** to only five independent elements," *Geophys. J. Int.*, **107**, 703-707 (1991).
- 19 L. J. Pyrak-Nolte, L. R. Myer, and N. G. W. Cook, "Transmission of seismic waves across single natural fractures," *J. Geophys. Res.*, **95**, No. B6, 8617-8638 (1990).
- 20 R. H. Tatham, M. D. Matthews, K. K. Sekharan, C. J. Wade, and L. M. Liro, "A physical model study of shear-wave splitting and fracture intensity," *Geophysics*, **57**, No. 4, 647-652 (1992).