

THE EFFECTIVE MODEL OF A STRATIFIED SOLID-FLUID MEDIUM AS A SPECIAL CASE OF THE BIOT MODEL

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The Biot model is compared with the effective model of a stratified periodic elastic-fluid medium, and the parameters of the transversally isotropic Biot model that turn it into the effective model are found. In the course of this comparison, some intermediate models, which are generalizations of the effective model and particular cases of the Biot model, are considered. For all of the models, the wave fronts excited by a point source are determined. The distinguishing feature of wave fronts in the intermediate models is the occurrence of double loops on some of them. Bibliography: 15 titles.

Wave propagation in fractured and porous media is commonly investigated on the basis of effective models because it is impossible to solve the problem of wave diffraction on one or several fractures or pores of arbitrary shape. In this paper, two such effective models are considered: the Biot model [1, 2] and the model of a stratified solid-fluid medium [2–5]. Both of them are two-phase models, and they are applicable only if pores and fractures are rather densely distributed. Moreover, both models have a high-frequency limitation, which is irrelevant to real-world seismic media.

The main conclusion of this paper is that the effective model of a stratified elastic-fluid medium is a particular case of the Biot model. To some extent, this conclusion justifies the Biot model, which is constructed on the basis of hypotheses, whereas the effective model of a stratified elastic-fluid medium corresponds to the asymptotics of a wave field. In order to establish a relationship between the two models, we consider intermediate models (media), which are generalizations of the effective model of a stratified elastic-fluid medium and particular cases of the Biot model. The wave fronts in the intermediate and Biot models and, especially, double loops on these fronts are of special interest.

§1. BIOT'S MODEL

Consider the homogeneous Biot model, in which the saturating fluid is isotropic but the elastic frame is transversally isotropic, and its anisotropy axis coincides with the z -axis. In this medium, the tortuosity tensor, defined by the location and shapes of pores, is assumed diagonal $[\alpha_1, \alpha_1, \alpha_2]$, so that the tortuosity in the xy plane is independent of direction. In accordance with [1, 2], such a medium has two phases, and with every point of this medium two displacement vectors and two stress tensors, which are related to the fluid (first) and solid (second) phases, are associated. It should be noted that both phases occupy connected interpenetrating regions. Isolated pores filled with the fluid are referred to the solid phase. Solid particles completely surrounded by the fluid are referred to the fluid phase. The introduction of two phases is related to the occurrence of both the elastic and fluid media in a small neighborhood of every point. For these two media, we can find the displacements $u_x^{(1)}, u_y^{(1)}, u_z^{(1)}$ and the stress $\tau^{(1)}$ averaged upon the fluid phase, and also the displacements $u_x^{(2)}, u_y^{(2)}, u_z^{(2)}$ and the stresses $\tau_{xx}^{(2)}, \tau_{yy}^{(2)}, \tau_{zz}^{(2)}, \tau_{yz}^{(2)}, \tau_{xz}^{(2)}, \tau_{xy}^{(2)}$ averaged upon the solid phase. These quantities completely characterize displacements and stresses at every point of the two-phase Biot medium. In order to take into account the fact that each of the phases occupies only a part of the volume of the porous medium, we also consider stresses averaged upon the total volume of a neighborhood of a point rather than upon the volumes of each of the phases. These stresses averaged upon the total volume are expressed by the equalities

$$\sigma_{ij}^{(2)} = (1 - \varepsilon)\tau_{ij}^{(2)}, \quad \sigma^{(1)} = \varepsilon\tau^{(1)} \quad (i, j = x, y, z), \quad (1.1)$$

in which porosity ε specifies the part of the volume that is occupied by the fluid phase. We should note that, apparently, in the papers of M. Biot [1, 2] only the case of small porosity $\varepsilon \ll 1$ is considered because

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the stresses $\sigma_{ij}^{(2)}$ and $\tau_{ij}^{(2)}$ are not distinguished. Actually, they must be distinguished because both phases occur on equal terms. Furthermore, the distinction between $\sigma_{ij}^{(2)}$ and $\tau_{ij}^{(2)}$ is needed to establish a relation between the effective model of a solid-fluid medium and the Biot model. Formulas analogous to relations (1.1) appeared in [6].

The stresses averaged upon the total volume and the displacements averaged upon each of the phases are interrelated via the equations of Hooke's law:

$$\begin{aligned}
 \sigma_{xx}^{(2)} &= P \frac{\partial u_x^{(2)}}{\partial x} + A \frac{\partial u_y^{(2)}}{\partial y} + F \frac{\partial u_z^{(2)}}{\partial z} + M\theta, \\
 \sigma_{yy}^{(2)} &= A \frac{\partial u_x^{(2)}}{\partial x} + P \frac{\partial u_y^{(2)}}{\partial y} + F \frac{\partial u_z^{(2)}}{\partial z} + M\theta, \\
 \sigma_{zz}^{(2)} &= F \frac{\partial u_x^{(2)}}{\partial x} + F \frac{\partial u_y^{(2)}}{\partial y} + C \frac{\partial u_z^{(2)}}{\partial z} + Q\theta, \\
 \sigma^{(1)} &= M \frac{\partial u_x^{(2)}}{\partial x} + M \frac{\partial u_y^{(2)}}{\partial y} + Q \frac{\partial u_z^{(2)}}{\partial z} + R\theta, \\
 \sigma_{yz}^{(2)} &= L \left(\frac{\partial u_y^{(2)}}{\partial z} + \frac{\partial u_z^{(2)}}{\partial y} \right), \quad \sigma_{xz}^{(2)} = L \left(\frac{\partial u_x^{(2)}}{\partial z} + \frac{\partial u_z^{(2)}}{\partial x} \right), \\
 \sigma_{xy}^{(2)} &= N \left(\frac{\partial u_x^{(2)}}{\partial y} + \frac{\partial u_y^{(2)}}{\partial x} \right), \quad \theta = \frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} + \frac{\partial u_z^{(1)}}{\partial z}
 \end{aligned} \tag{1.2}$$

and the equations of a continuous medium:

$$\begin{aligned}
 \frac{\partial \sigma_{xx}^{(2)}}{\partial x} + \frac{\partial \sigma_{xy}^{(2)}}{\partial y} + \frac{\partial \sigma_{xz}^{(2)}}{\partial z} &= \rho'_{11} \frac{\partial^2 u_x^{(2)}}{\partial t^2} + \rho'_{12} \frac{\partial^2 u_z^{(1)}}{\partial t^2}, \\
 \frac{\partial \sigma_{xy}^{(2)}}{\partial x} + \frac{\partial \sigma_{yy}^{(2)}}{\partial y} + \frac{\partial \sigma_{yz}^{(2)}}{\partial z} &= \rho'_{11} \frac{\partial^2 u_y^{(2)}}{\partial t^2} + \rho'_{12} \frac{\partial^2 u_y^{(1)}}{\partial t^2}, \\
 \frac{\partial \sigma_{xz}^{(2)}}{\partial x} + \frac{\partial \sigma_{yz}^{(2)}}{\partial y} + \frac{\partial \sigma_{zz}^{(2)}}{\partial z} &= \rho''_{11} \frac{\partial^2 u_z^{(2)}}{\partial t^2} + \rho''_{12} \frac{\partial^2 u_z^{(1)}}{\partial t^2}, \\
 \frac{\partial \sigma^{(1)}}{\partial x} &= \rho'_{12} \frac{\partial^2 u_x^{(2)}}{\partial t^2} + \rho'_{22} \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \\
 \frac{\partial \sigma^{(1)}}{\partial y} &= \rho'_{12} \frac{\partial^2 u_y^{(2)}}{\partial t^2} + \rho'_{22} \frac{\partial^2 u_y^{(1)}}{\partial t^2}, \\
 \frac{\partial \sigma^{(1)}}{\partial z} &= \rho''_{12} \frac{\partial^2 u_z^{(2)}}{\partial t^2} + \rho''_{22} \frac{\partial^2 u_z^{(1)}}{\partial t^2}.
 \end{aligned} \tag{1.3}$$

The coefficients in Eqs. (1.2) and (1.3) specify the given Biot medium. The matrix formed by the coefficients occurring in relations (1.2) is positive definite, and $P = A + 2N$. In the general case, Biot's theory treats these coefficients as indefinite and indicates experiments in the course of which they can be measured [7]. Explicit expressions for these coefficients are given only in some special cases. Another group of parameters is determined by the equalities

$$\begin{aligned}
 \rho'_{11} &= \rho_s(1 - \varepsilon) + \varepsilon \rho_f(\alpha_1 - 1), & \rho''_{11} &= \rho_s(1 - \varepsilon) + \varepsilon \rho_f(\alpha_2 - 1), \\
 \rho'_{12} &= -\rho_f \varepsilon(\alpha_1 - 1), & \rho''_{12} &= -\rho_f \varepsilon(\alpha_2 - 1), \\
 \rho'_{22} &= \rho_f \varepsilon \alpha_1, & \rho''_{22} &= \rho_f \varepsilon \alpha_2,
 \end{aligned} \tag{1.4}$$

where ρ_s and ρ_f are the densities of the solid and fluid phases, respectively. Thus, the Biot model considered is characterized by 13 parameters: $A, F, M, C, Q, R, L, N, \rho_s, \rho_f, \alpha_1, \alpha_2$, and ε .

The densities of kinetic and potential energies corresponding to the model described by Eqs. (1.2), (1.3) are as follows:

$$\begin{aligned}
W_k &= \frac{1}{2} \left\{ \rho'_{11} \left[\left(\frac{\partial u_x^{(2)}}{\partial t} \right)^2 + \left(\frac{\partial u_y^{(2)}}{\partial t} \right)^2 \right] + 2\rho'_{12} \left(\frac{\partial u_x^{(2)}}{\partial t} \frac{\partial u_x^{(1)}}{\partial t} + \frac{\partial u_y^{(2)}}{\partial t} \frac{\partial u_y^{(1)}}{\partial t} \right) + \right. \\
&\quad \left. + \rho'_{22} \left[\left(\frac{\partial u_x^{(1)}}{\partial t} \right)^2 + \left(\frac{\partial u_y^{(1)}}{\partial t} \right)^2 \right] + \rho''_{11} \left(\frac{\partial u_z^{(2)}}{\partial t} \right)^2 + 2\rho''_{12} \frac{\partial u_z^{(2)}}{\partial t} \frac{\partial u_z^{(1)}}{\partial t} + \rho''_{22} \left(\frac{\partial u_z^{(1)}}{\partial t} \right)^2 \right\}, \\
W_p &= \frac{1}{2} \left\{ P \left[\left(\frac{\partial u_x^{(2)}}{\partial x} \right)^2 + \left(\frac{\partial u_y^{(2)}}{\partial y} \right)^2 \right] + 2A \frac{\partial u_x^{(2)}}{\partial x} \frac{\partial u_y^{(2)}}{\partial y} + 2F \left(\frac{\partial u_x^{(2)}}{\partial x} + \frac{\partial u_y^{(2)}}{\partial y} \right) \frac{\partial u_z^{(2)}}{\partial z} + \right. \\
&\quad + 2M \left(\frac{\partial u_x^{(2)}}{\partial x} + \frac{\partial u_y^{(2)}}{\partial y} \right) \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} + \frac{\partial u_z^{(1)}}{\partial z} \right) + C \left(\frac{\partial u_z^{(2)}}{\partial z} \right)^2 + \\
&\quad + 2Q \frac{\partial u_z^{(2)}}{\partial z} \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} + \frac{\partial u_z^{(1)}}{\partial z} \right) + R \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} + \frac{\partial u_z^{(1)}}{\partial z} \right)^2 + \\
&\quad \left. + L \left[\left(\frac{\partial u_y^{(2)}}{\partial z} + \frac{\partial u_z^{(2)}}{\partial y} \right)^2 + \left(\frac{\partial u_x^{(2)}}{\partial z} + \frac{\partial u_z^{(2)}}{\partial x} \right)^2 \right] + N \left(\frac{\partial u_x^{(2)}}{\partial y} + \frac{\partial u_y^{(2)}}{\partial x} \right)^2 \right\}. \tag{1.5}
\end{aligned}$$

The expression for W_p is a positive definite quadratic form in the arguments $\partial u_x^{(2)}/\partial x$, $\partial u_y^{(2)}/\partial y$, $\partial u_z^{(2)}/\partial z$, $\partial u_x^{(1)}/\partial x + \partial u_y^{(1)}/\partial y + \partial u_z^{(1)}/\partial z$, $\partial u_y^{(2)}/\partial z + \partial u_z^{(2)}/\partial y$, $\partial u_x^{(2)}/\partial z + \partial u_z^{(2)}/\partial x$, $\partial u_x^{(2)}/\partial y + \partial u_y^{(2)}/\partial x$, and θ because the coefficients in (1.2) form a positive definite matrix. The expression for W_k is a sum of three positive definite forms in the arguments $\partial u_x^{(2)}/\partial t$, $\partial u_x^{(1)}/\partial t$; $\partial u_y^{(2)}/\partial t$, $\partial u_y^{(1)}/\partial t$, and $\partial u_z^{(2)}/\partial t$, $\partial u_z^{(1)}/\partial t$, respectively. This can easily be verified by writing, on the basis of equality (1.4), the relations

$$\begin{aligned}
\rho'_{11}\rho'_{22} - \rho'^2_{12} &= \rho_s \rho_f \varepsilon (1 - \varepsilon) \alpha_1 + \rho_f^2 \varepsilon^2 (\alpha_1 - 1), \\
\rho''_{11}\rho''_{22} - \rho''^2_{12} &= \rho_s \rho_f \varepsilon (1 - \varepsilon) \alpha_2 + \rho_f^2 \varepsilon^2 (\alpha_2 - 1), \tag{1.6}
\end{aligned}$$

whose right-hand sides are positive because the tortuosities α_1 and α_2 lie in the interval $[1, \infty)$ [2].

The purpose of this paper is to establish a relationship between the given Biot model and the effective model of an elastic-fluid medium. Such a relationship will be determined in two two-dimensional cases similar to the cases of propagation of $P - SV$ and SH waves in an elastic medium. In considering waves of the first type, we will investigate a sequence of intermediate models, in which every next model is a particular case of the previous one.

§2. MEDIUM 1

Let the displacement field in the Biot model not contain the components $u_y^{(1)}$ and $u_y^{(2)}$ and let the fields of stresses and displacements be independent of the coordinate y . These assumptions lead to the elimination of the second, fifth, and seventh equations in (1.2) and of the second and fifth equations in (1.3). In the remaining equations, the terms involving $u_y^{(1)}$, $u_y^{(2)}$, σ_{xy} , and σ_{yz} vanish. After these simplifications the system of equations (1.2) and (1.3) describes the propagation of the two-dimensional waves $P_2 - P_1 - SV$ in the Biot medium, which is analogous to the propagation of the waves $P - SV$ in an elastic medium.

Another assumption is that, in the medium considered, the shear stress $\sigma_{xz}^{(2)}$ vanishes and, in addition,

$$L = 0. \tag{2.1}$$

Assumption (2.1) is justified if the elastic-fluid interfaces on which the tangential stresses vanish are densely distributed and located close to each other. The Biot model with $L = 0$ is referred to as medium 1.

In order to study wave fronts excited in this medium by a point source acting at the origin of coordinates since $t = 0$, we express the displacements $u_x^{(1)}$, $u_x^{(2)}$, $u_z^{(1)}$, $u_z^{(2)}$ and the stresses $\sigma^{(1)}$, $\sigma_{xx}^{(2)}$, $\sigma_{zz}^{(2)}$ for $z > 0$ by

integrals of the forms

$$\begin{aligned} u_x^{(1)} &= \int_0^\infty \frac{e^{ikx} dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} U_x^{(1)} e^{k(t\eta-z\beta)} d\eta, \\ \sigma^{(1)} &= \int_0^\infty \frac{ke^{ikx} dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} S^{(1)} e^{k(t\eta-z\beta)} d\eta, \end{aligned} \quad (2.2)$$

respectively. On substitution of such integrals in the first, third, and fourth equations in (1.2) and the first, third, fourth, and sixth equations in (1.3), the latter equations are replaced by a homogeneous linear system of seven equations in the unknowns $U_x^{(1)}$, $U_z^{(1)}$, $U_x^{(2)}$, $U_z^{(2)}$, $S^{(1)}$, $S_{xx}^{(2)}$, $S_{zz}^{(2)}$, which are the transforms of the displacements and stresses. This system has a nonzero solution only if its determinant is zero. After calculating the determinant of the system, this condition leads to the biquadratic equation

$$\begin{aligned} &\beta^4 \left[\eta^2 \delta' \left| \begin{smallmatrix} C & Q \\ Q & R \end{smallmatrix} \right| + \rho'_{22} \Delta \right] - \beta^2 \left[\eta^4 \delta' (\rho''_{11} R + \rho''_{22} C - 2\rho''_{12} Q) + \right. \\ &\quad \left. + \eta^2 \left(\rho'_{11} \rho''_{22} \left| \begin{smallmatrix} C & Q \\ Q & R \end{smallmatrix} \right| + \rho'_{22} \rho''_{11} \left| \begin{smallmatrix} P & M \\ M & R \end{smallmatrix} \right| + \rho'_{22} \rho''_{22} \left| \begin{smallmatrix} P & F \\ F & C \end{smallmatrix} \right| - \right. \right. \\ &\quad \left. \left. - 2\rho'_{12} \rho''_{12} \left| \begin{smallmatrix} F & M \\ Q & R \end{smallmatrix} \right| - 2\rho'_{22} \rho''_{12} \left| \begin{smallmatrix} P & F \\ M & Q \end{smallmatrix} \right| + 2\rho'_{12} \rho''_{22} \left| \begin{smallmatrix} F & M \\ C & Q \end{smallmatrix} \right| \right) + \rho''_{22} \Delta \right] + \\ &\quad \left. + \delta'' \eta^2 \left[\delta' \eta^4 + \eta^2 (P\rho'_{22} + R\rho'_{11} - 2M\rho'_{12}) + \left| \begin{smallmatrix} P & M \\ M & R \end{smallmatrix} \right| \right] = 0, \end{aligned} \quad (2.3)$$

where we use the notation

$$\begin{aligned} \delta' &= \rho'_{11} \rho'_{22} - \rho'^2_{12}, & \Delta &= \left| \begin{smallmatrix} P & F & M \\ F & C & Q \\ M & Q & R \end{smallmatrix} \right|. \\ \delta'' &= \rho''_{11} \rho''_{22} - \rho''^2_{12}, \end{aligned} \quad (2.4)$$

The functions $\beta_1(\eta)$ and $\beta_2(\eta)$ that are solutions of Eqs. (2.3) determine the fronts of waves propagating from a point source acting at the origin of coordinates. In order to construct these wave fronts, it is necessary to find the intervals on the positive part of the imaginary axis of the η plane where the quantities $\beta_1(\eta)$ and $\beta_2(\eta)$ are pure imaginary, and to consider the functions $\bar{\beta}_1(\tau) = \text{Im} \beta_1(i\tau)$ and $\bar{\beta}_2(\tau) = \text{Im} \beta_2(i\tau)$ on these intervals. In accordance with [8], the wave fronts are parametrically defined by the formulas

$$x = t \left(\tau - \frac{\bar{\beta}_i(\tau)}{\bar{\beta}'_i(\tau)} \right), \quad z = \frac{t}{\bar{\beta}'_i(\tau)} \quad (i = 1, 2), \quad (2.5)$$

and every value of τ from the intervals mentioned above determines a point on the front. From (2.5) and the condition of the finiteness of the velocity of wave propagation it follows that

$$\bar{\beta}'_i(\tau) > 0 \quad (2.6)$$

at all points of the intervals in question. The function $\bar{\beta}_i(\tau)$ determines the angle $(\widehat{n, z})$ between the normal to the front and the z -axis in accordance with the formula

$$\bar{\beta}_i(\tau) = \text{ctg}(\widehat{n, z}). \quad (2.7)$$

In view of inequality (2.6) and relation (2.7), this angle monotonically decreases as τ increases; in addition, the direction of the normal steadily turns toward the z -axis. The convexity or concavity of the fronts is determined by the second derivatives $\bar{\beta}''_i(\tau)$ via the equality

$$\frac{d^2 x}{dz^2} = \frac{(\bar{\beta}'_i)^3}{t \bar{\beta}''_i}, \quad (2.8)$$

obtained in [9]. The front is convex if $\bar{\beta}_i'' < 0$ and concave if $\bar{\beta}_i'' > 0$. At points of inflection we have $\bar{\beta}_i''(\tau) = 0$. The connection between the functions $\bar{\beta}_i$ and wave fronts expressed by Eqs. (2.5)–(2.8) is rather general and applies to a certain class of media, which includes, at least, transversally isotropic porous and nonporous media.

In order to construct wave fronts excited by a point source, we will find the intervals where the functions $\beta_i(i\tau)$ are pure imaginary and plot the graphs of the functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$. To this end, using Vieta's theorem, we write the relation

$$\bar{\beta}_1^2(\tau)\bar{\beta}_2^2(\tau) = \frac{\delta''\tau^2[\delta'\tau^4 - \tau^2(P\rho'_{22} + R\rho'_{11} - 2M\rho'_{12}) + PR - M^2]}{\tau^2\delta'(CR - Q^2) - \rho'_{22}\Delta}, \quad (2.9)$$

which implies that the product $\bar{\beta}_1(\tau)\bar{\beta}_2(\tau)$ has one zero root and the roots τ_1 and τ_2 given by the relations

$$\tau_{1,2}^2 = \left[P\rho'_{22} + R\rho'_{11} - 2M\rho'_{12} \pm \sqrt{(P\rho'_{22} + R\rho'_{11} - 2M\rho'_{12})^2 - 4\delta'(PR - M^2)} \right] / (2\delta'), \quad (2.10)$$

whereas

$$\tau_3 = \sqrt{\rho'_{22}\Delta / [\delta'(CR - Q^2)]} \quad (2.11)$$

is a pole of this product.

For all possible values of the parameters, the quantities occurring in equalities (2.10) and (2.11) satisfy the inequalities

$$\begin{aligned} P\rho'_{22} + R\rho'_{11} - 2M\rho'_{12} &> 0, \\ (P\rho'_{22} + R\rho'_{11} - 2M\rho'_{12})^2 &> 4\delta'(PR - M^2), \\ PR - M^2 &> 0, \quad CR - Q^2 > 0, \quad \delta' > 0. \end{aligned} \quad (2.12)$$

They are proved by using the positive definiteness of the matrix of Hooke's law (1.2) and formulas (2.4), (1.6). From inequalities (2.12) it follows that $\tau_1^2 > 0$, $\tau_2^2 > 0$, and $\tau_3^2 > 0$. The quantities τ_1 , τ_2 , and τ_3 are interrelated as follows:

$$\tau_1 > \tau_2, \quad \tau_1 > \tau_3. \quad (2.13)$$

The first inequality in (2.13) is trivial, whereas to prove the second one we first find the maximum value of τ_3^2 as a function of F and then square both parts of the inequality obtained in this way.

The semiaxis $\tau \geq 0$ is divided into four intervals by the points $\min(\tau_2, \tau_3)$, $\max(\tau_2, \tau_3)$, and τ_1 . In these intervals, the functions $\bar{\beta}_1^2(\tau)$ and $\bar{\beta}_2^2(\tau)$ may be either positive or negative, whereas within each of the intervals they preserve their signs. From equality (2.9) it follows that

$$\begin{aligned} \bar{\beta}_1^2\bar{\beta}_2^2 &< 0 \text{ if either } 0 < \tau < \min(\tau_2, \tau_3) \text{ or } \max(\tau_2, \tau_3) < \tau < \tau_1, \\ \bar{\beta}_1^2\bar{\beta}_2^2 &> 0 \text{ if either } \min(\tau_2, \tau_3) < \tau < \max(\tau_2, \tau_3) \text{ or } \tau > \tau_1. \end{aligned} \quad (2.14)$$

In order to determine the signs of the functions $\bar{\beta}_1^2$ and $\bar{\beta}_2^2$ in the intervals in question, we analyze the solutions of Eqs. (2.3) in the vicinity of the points $\tau = 0$ and $\tau = \infty$ and prove that the graphs of the functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$ have no points in common. For small τ , the approximate equalities

$$\bar{\beta}_1^2 = -\rho''_{22}/\rho'_{22}, \quad \bar{\beta}_2^2 = -\tau^2/\tau_6^2$$

are valid, where

$$\tau_6^2 = \rho''_{22}\Delta / [\delta''(PR - M^2)]. \quad (2.16)$$

For large τ , the asymptotic relations

$$\bar{\beta}_{1,2}^2 = \tau^2/\tau_{5,4}^2 \quad (2.17)$$

are satisfied. Here, $\tau_{5,4}^2$ are the roots of the equation

$$\delta''\tau^4 - (C\rho_{22}'' + R\rho_{11}'' - 2Q\rho_{12}'')\tau^2 + CR - Q^2 = 0, \quad (2.18)$$

which are provided by the formulas

$$\tau_{4,5}^2 = \left[C\rho_{22}'' + R\rho_{11}'' - 2Q\rho_{12}'' \pm \sqrt{(C\rho_{22}'' + R\rho_{11}'' - 2Q\rho_{12}'')^2 - 4\delta''(CR - Q^2)} \right] / (2\delta''). \quad (2.19)$$

The quantities τ_4^2 , τ_5^2 , and τ_6^2 are positive by virtue of (2.12) and the inequalities

$$C\rho_{22}'' + R\rho_{11}'' - 2Q\rho_{12}'' > 0, \quad (C\rho_{22}'' + R\rho_{11}'' - 2Q\rho_{12}'')^2 > 4\delta''(CR - Q^2), \quad \delta'' > 0. \quad (2.20)$$

Inequalities (2.20) are proved using the positive definiteness of the matrix of Hooke's law (1.2) and formulas (2.4) and (1.6). If we interchange the positions of the x - and z -axes and express the displacements and stresses by integrals of the type (2.2) as previously, then the product $\bar{\beta}_1^2\bar{\beta}_2^2$ of the new functions has the roots 0, τ_4 , τ_5 and the pole τ_6 . Since these two choices of the axes are completely analogous, the inequalities

$$\tau_4 > \tau_5, \quad \tau_4 > \tau_6, \quad (2.21)$$

similar to (2.13), are valid.

In order to prove that the graphs of the functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$ have no points in common in the region $\bar{\beta}_1 > 0$, $\bar{\beta}_2 > 0$, we assume that such a point τ_0 exists. At this point the discriminant of Eq. (2.3) vanishes, and therefore the derivatives $\bar{\beta}_1'(\tau_0)$ and $\bar{\beta}_2'(\tau_0)$ are infinite and have opposite signs. Since this conclusion contradicts condition (2.6), the assumption on the existence of common points of the graphs of the functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$ is wrong.

Now we examine the character of the variation of the functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$ in the four intervals of the semiaxis $\tau \geq 0$ and represent the results of these investigations in Figs. 1 and 2. In accordance with relations (2.4), in the interval $0 \leq \tau < \min(\tau_2, \tau_3)$ the quantity $\bar{\beta}_1(\tau)$ is pure imaginary, but the function $\bar{\beta}_2(\tau)$ increases from zero, and its graph at the point $\tau = 0$ is tangent to the straight line $\bar{\beta} = \tau/\tau_6$. If $\tau_3 < \tau_2$, then at the upper endpoint of this interval we have $\bar{\beta}_2(\tau_3) = \infty$. Under this condition, both functions considered are pure imaginary in the second interval $\tau_3 < \tau < \tau_2$. If, on the contrary, $\tau_2 > \tau_3$, then both functions are real in the second interval. The function $\bar{\beta}_1(\tau)$ increases from zero, and $\bar{\beta}_2(\tau)$ approaches an infinite value at $\tau = \tau_3$. In the third interval, $\max(\tau_2, \tau_3) < \tau \leq \tau_1$, the function $\bar{\beta}_1(\tau)$ is real and increasing, whereas $\bar{\beta}_2(\tau)$ is pure imaginary. Finally, in the fourth interval, $\tau > \tau_1$, both functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$ are real, and they increase and approach the asymptotes $\bar{\beta} = \tau/\tau_5$ and $\bar{\beta} = \tau/\tau_4$, respectively. The approximate graphs of the functions $\bar{\beta}_{1,2}(\tau)$ constructed on the basis of the above analysis are represented in Figs. 1 ($\tau_2 > \tau_3, \tau_5 > \tau_6$) and 2 ($\tau_3 > \tau_2, \tau_5 > \tau_6$). In both cases, the graphs consist of three branches. The branch that issues out of the point $(0, \tau_1)$ is always convex, whereas each of the two remaining branches may have two points of inflection.

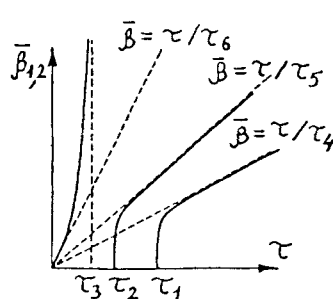


FIG. 1

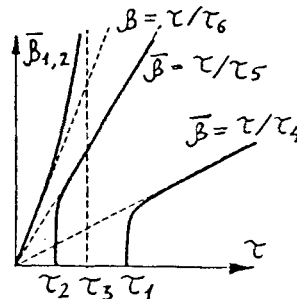


FIG. 2

In order to construct the wave fronts, based on formulas (2.5) and (2.7), we determine to what parts of the fronts neighborhoods of the origins and endpoints of each of the branches of $\bar{\beta}_{1,2}(\tau)$ correspond. In accordance with the second equality in (2.15), a neighborhood of the point $x = 0, z = \tau_6 t$ on the front in which the normal is orthogonal to the z -axis corresponds to small τ 's. To the opposite end of this curve there corresponds the point $x = \tau_3 t, z = 0$, at which the normal is parallel to the z -axis. The last conclusion is based on the easily verified relations $\bar{\beta}(\tau_3) = \infty, \bar{\beta}'(\tau_3) = \infty$, and $\frac{\bar{\beta}(\tau_3)}{\bar{\beta}'(\tau_3)} = 0$. The points $\tau = \tau_2 t, z = 0$ and $x = \tau_1 t, z = 0$, at which the normals are perpendicular to the z -axis, correspond to the points $\tau = \tau_2$ and $\tau = \tau_1$, where $\bar{\beta}_1(\tau_2) = 0, \bar{\beta}_2(\tau_1) = 0, \bar{\beta}'_1(\tau_2) = \infty$, and $\bar{\beta}'_2(\tau_1) = \infty$. Finally, to the neighborhoods of the infinitely distant points at which $\bar{\beta}_1 = \tau/\tau_5$ and $\bar{\beta}_2 = \tau/\tau_4$ there correspond some parts of the fronts containing the points $x = 0, z = \tau_5 t$ and $x = 0, z = \tau_4 t$, at which the normals are parallel to the z -axis. If we connect points of the fronts that correspond to one and the same branch of the graph by curves on which the angle between the normal and the z -axis varies in a monotone way, then we obtain an approximate location of the fronts. In Figs. 3-6, one can see the wave fronts that correspond to the four cases, in which loops are absent: (1) $\tau_2 > \tau_3, \tau_5 > \tau_6$; (2) $\tau_2 > \tau_3, \tau_6 > \tau_5$; (3) $\tau_2 < \tau_3, \tau_5 > \tau_6$; (4) $\tau_2 < \tau_3, \tau_5 < \tau_6$.

In each of the four cases, three waves propagate in medium 1. The fronts with velocities τ_1, τ_4 and τ_2, τ_5 along the axes belong to the quasilongitudinal waves P_1 and P_2 associated with the elastic and fluid phases, respectively. The third front, which is tangent to the coordinate axes, is obtained from the loop of the quasitransverse wave SV in the Biot model as $L \rightarrow 0$. Under this passage to the limit, the loop increases in size, its cusps emerge on the coordinate axes, the concave part of the front still exists and even expands, whereas its convex parts merge on the coordinate axes with the corresponding parts arriving from the adjacent quadrants. A similar transformation of shear-wave fronts was established in [8] for a transversally isotropic elastic medium in the case where the shear modulus c_{44} tends to zero.

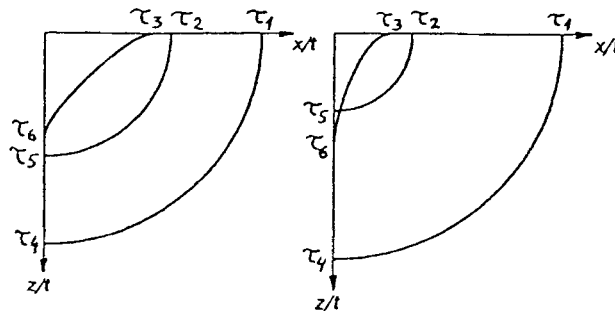


FIG. 3

FIG. 4

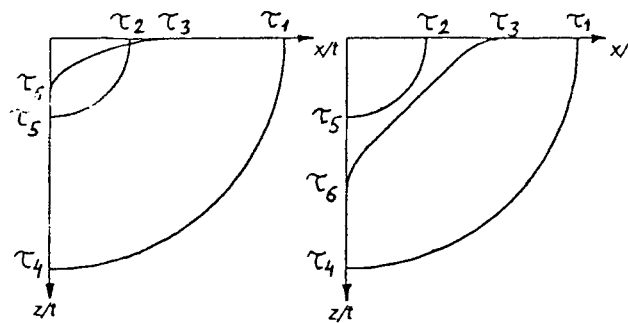


FIG. 5

FIG. 6

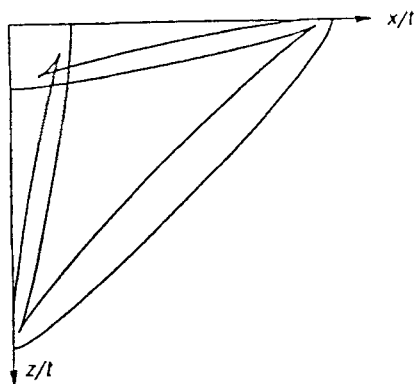


FIG. 7

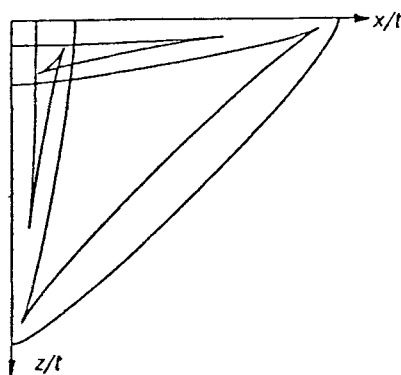


FIG. 8

The fronts of the waves P_2 and SV represented in Figs. 3–6 may prove to be more sophisticated because of the presence of loops. Similarly to fronts in anisotropic elastic media, the front of the wave P_2 may have a concave loop, whereas the front of the wave SV may contain an abnormal convex loop located behind the front (see Fig. 7). In that case, points of inflection on the graphs of the functions $\bar{\beta}_1(\tau)$ and $\bar{\beta}_2(\tau)$ correspond to the cusps of the wave fronts. On the branches that issue out of the points $(0, 0)$ and $(\tau_2, 0)$, either two points of inflection occur or there are no such points at all. In the case of medium 1, the number of cusps on every front cannot exceed 2. This conclusion is based on examining refraction curves, which cannot be intersected by an arbitrary straight line at more than six points. Here, 6 is the order of Cristoffel's equation (2.3) with respect to η . It should be pointed out that the leading front is always convex, has no loops, and does not intersect other wave fronts. This conclusion corresponds to Huygens and Fermat principles and is confirmed by all investigations.

We emphasize that double loops on the front of the wave SV in the Biot model may arise in the case $L \neq 0$. Such loops are shown in Fig. 8, where $P = C = 32.0$ GPa, $F = 4.3$ GPa, $M = Q = 1.1$ GPa, $R = 0.4$ GPa, $L = 0.2$ GPa, $\alpha_1 = \alpha_2 = 3$, $\rho_s = 2.7$ g/cm³, $\rho_f = 1.0$ g/cm³, $\varepsilon = 0.2$. The existence of double loops does not contradict the rule limiting the maximum number of intersections of a refraction curve with an arbitrary straight line. Upon passing to the limit as $L \rightarrow 0$, a double loop turns into a simple loop with convex boundary, whereas other parameters are preserved (see Fig. 7).

§3. MEDIA 2 AND 3

Now from medium 1 we pass to the stratified elastic-fluid media, in which the equalities

$$\tau_{zz}^{(2)} = \tau^{(1)}, \quad (3.1)$$

$$\alpha_1 = 1, \quad \alpha_2 = \infty \quad (3.2)$$

are satisfied. First we consider medium 2, which is a particular case of medium 1, in which only relation (3.1) is assumed. It is fulfilled on the interfaces $z = \text{const}$ between the fluid and elastic media. For this relation to be satisfied in the whole medium, it is necessary that the elastic-fluid interfaces be densely distributed and parallel to the plane $z = \text{const}$. In accordance with (1.1), relation (3.1) leads to the equality

$$\varepsilon \sigma_{zz}^{(2)} = (1 - \varepsilon) \sigma^{(1)}, \quad (3.3)$$

which, along with the third and fourth equations in (1.2), implies the formulas

$$M = \varepsilon F / (1 - \varepsilon), \quad Q = \varepsilon C / (1 - \varepsilon), \quad R = \varepsilon^2 C / (1 - \varepsilon)^2. \quad (3.4)$$

On substitution of relations (3.4) into the first and third equations in (1.2), the latter equations (because they have no terms containing $\frac{\partial u_z}{\partial y}$) are replaced by the new equations of Hooke's law

$$\begin{aligned} \sigma_{xz}^{(2)} &= P \frac{\partial u_x^{(2)}}{\partial x} + \frac{F}{1 - \varepsilon} \left[(1 - \varepsilon) \frac{\partial u_z^{(2)}}{\partial z} + \varepsilon \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_z^{(1)}}{\partial z} \right) \right], \\ \sigma_{zz}^{(2)} &= F \frac{\partial u_x^{(2)}}{\partial x} + \frac{C}{1 - \varepsilon} \left[(1 - \varepsilon) \frac{\partial u_z^{(2)}}{\partial z} + \varepsilon \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_z^{(1)}}{\partial z} \right) \right]. \end{aligned} \quad (3.5)$$

The equations of a continuous medium are not affected by assumption (3.1).

In order to construct wave fronts in medium 2, it is convenient to utilize the function β , as in the case of medium 1. The expression

$$\begin{aligned} \beta^2 &= \delta'' \{ \delta' (1 - \varepsilon)^2 \tau^4 - \tau^2 [P \rho'_{22} (1 - \varepsilon)^2 - 2\varepsilon (1 - \varepsilon) F \rho'_{12} + C \rho'_{11} \varepsilon^2] + \\ &+ \varepsilon^2 (PC - F^2) \} / \{ [\varepsilon^2 \rho''_{11} - 2\varepsilon (1 - \varepsilon) \rho''_{12} + (1 - \varepsilon)^2 \rho''_{22}] [C \delta' \tau^2 - \rho'_{22} (PC - F^2)] \} \end{aligned} \quad (3.6)$$

for its square may be derived in two ways. The first way consists of substituting relations (3.4) into Eq. (2.3) and taking into account the equalities $\Delta = 0$ and $CR = Q^2$. Another possibility is to represent displacements and stresses by integrals of the type (2.2) and to substitute the latter into Eqs. (3.5) and (1.3). Then Eq. (3.6) arises as the condition of the existence of a nontrivial solution of the resulting algebraic system of equations in 6 unknowns.

As in §2, the function $\beta(\eta)$ is replaced by the real function $\bar{\beta}(\tau)$, where $\tau = \text{Im } \eta$, $\text{Re } \eta = 0$, $\bar{\beta} = \text{Im } \beta$, and $\text{Re } \beta = 0$. In accordance with (3.6), the function $\bar{\beta}(\tau)$ vanishes if

$$\begin{aligned} \tau_{1,2}^2 &= \left\{ \pm \sqrt{[P \rho'_{22} (1 - \varepsilon)^2 - 2\varepsilon (1 - \varepsilon) F \rho'_{12} + C \rho'_{11} \varepsilon^2]^2 - 4\delta' \varepsilon^2 (1 - \varepsilon)^2 (PC - F^2) +} \right. \\ &\quad \left. + P \rho'_{22} (1 - \varepsilon)^2 - 2\varepsilon (1 - \varepsilon) F \rho'_{12} + C \rho'_{11} \varepsilon^2 \right\} / [2\delta' (1 - \varepsilon)^2] \end{aligned} \quad (3.7)$$

and becomes infinite if

$$\tau_3^2 = \rho'_{22} (PC - F^2) / (C \delta').$$

Relations (3.7) differ from (2.10) only in that the parameters have been changed in accordance with (3.4). In the case of medium 2, the quantities τ_1 , τ_2 , and τ_3 satisfy the inequalities

$$\tau_1 \geq \tau_2 \geq \tau_3. \quad (3.9)$$

In order to prove relations (3.9), we introduce the function

$$f(\tau) = \delta' (1 - \varepsilon)^2 \tau^4 - \tau^2 [P \rho'_{22} (1 - \varepsilon)^2 - 2\varepsilon (1 - \varepsilon) F \rho'_{12} + C \rho'_{11} \varepsilon^2] + \varepsilon^2 (PC - F^2),$$

which satisfies the inequalities

$$\begin{aligned} f(\tau) &\geq 0 && \text{if either } \tau \geq \tau_1 \text{ or } \tau \leq \tau_2, \\ f(\tau) &\leq 0 && \text{if } \tau_2 \leq \tau \leq \tau_1. \end{aligned}$$

Relations (3.9) hold because

$$f(\tau_3) = -\frac{\varepsilon(PC - F^2)}{\delta' C^2(1 - \varepsilon)^2} [\rho'_{22} F(1 - \varepsilon) - \rho'_{12} \varepsilon C]^2 \leq 0.$$

The quantities τ_4 , τ_5 , and τ_6 satisfy the equalities

$$\tau_4^2 = C[\varepsilon^2 \rho''_{11} - 2\varepsilon(1 - \varepsilon)\rho''_{12} + (1 - \varepsilon)\rho''_{22}] / [(1 - \varepsilon)^2 \delta''], \quad \tau_5^2 = \tau_6^2 = 0. \quad (3.10)$$

On the basis of the graphs of the functions $\bar{\beta}_{1,2}(\tau)$ in the case of medium 1, we construct the graph of the function $\bar{\beta}(\tau)$ by passing to the limit as

$$M \rightarrow \varepsilon F / (1 - \varepsilon), \quad Q \rightarrow \varepsilon C / (1 - \varepsilon), \quad R \rightarrow \varepsilon^2 C / (1 - \varepsilon)^2. \quad (3.11)$$

The left branches of the graphs shown in Figs. 1 and 2 come arbitrarily close to the ordinate axis because, under conditions (3.11), the slope $1/\tau_6$ of the tangent increases unlimitedly. In carrying out this passage to the limit, the asymptote of the middle branch with slope $1/\tau_5$ becomes vertical because $\tau_5 \rightarrow 0$. If M , Q , and R are close to the limit values but still not equal to them, then the middle branch itself issues out of the point $(\tau_2, 0)$, first rises up, then approaches the straight line $\tau = \tau_3$, moves along this straight line, and, finally, tends asymptotically to the line $\bar{\beta} = \tau/\tau_5$. This branch is shown in Fig. 9, in which we observe that the middle curve has two points of inflection, and, in accordance with §2, it may not have more of them. In passing to the limit (3.11), a part of the middle branch near the straight line $\tau = \tau_3$ unboundedly increases, whereas the part of this branch where the curve approaches its asymptote $\bar{\beta} = \tau/\tau_5$ goes to infinity. One of the inflection points also goes to infinity. After passage to the limit the middle branch issues out of the point $(\tau_2, 0)$, at which it is tangent to the line $\tau = \tau_2$, and then asymptotically tends to the line $\tau = \tau_3$. On the limit curve there is just one point of inflection (see Fig. 10). The right branch does not change significantly under this passage to the limit and remains convex. The limit location of the curves is shown in Fig. 10 and is consistent with formula (3.6).

In order to construct the wave fronts that are excited by a point source placed at the origin of coordinates, we use relations (2.5)–(2.8), as in §2. The point $x = \tau_2 t$, $z = 0$ of the front corresponds to the value $\bar{\beta}(\tau_2) = 0$. In the vicinity of this point, the front is convex, and the normal to it is perpendicular to the z -axis. With the value $\bar{\beta}(\tau_3) = \infty$ the point $x = \tau_3 t$, $z = 0$ is associated, in the vicinity of which the front is concave, and the normal is parallel to the z -axis. Between the points indicated in Fig. 10 there is a point of inflection, which corresponds to a cusp of the front (see Fig. 11). With the right branch, issuing out of the point $(\tau_1, 0)$, the leading front is associated, which is convex, has no loops, and does not intersect other fronts.

The wave fronts indicated in Fig. 11 completely correlate with the fronts of the two-phase effective model of a stratified elastic-fluid periodic medium. In order to explain how the fronts shown in Fig. 11 are obtained from those shown in Figs. 3–6, we plot the graphs in an intermediate case (see Fig. 9). The fronts corresponding to Fig. 9 are presented in Fig. 12. The middle front has a loop, the cusps of which correspond to the points of inflection on the middle branch in Fig. 9. As the parameters M , Q , and R approach their limiting values, the upper corner point comes closer to the x -axis, the concave front with velocities τ_3 , τ_6 and the part of the middle front between the upper corner point and the point $(0, \tau_5)$ emerge on the x -axis, where they merge with the corresponding fronts coming from the quadrant $x > 0$, $z < 0$. Upon these replacements, the fronts shown in Fig. 12 take the form indicated in Fig. 11.

Now consider medium 3, which is a particular case of medium 1 with additional conditions (3.2). Relations (3.2) first appeared in [11]. The equality $\alpha_1 = 1$ means that along the x -axis the phases are not

coupled, whereas the equality $\alpha_2 = \infty$ holds if both phases are closely related along the z -axis. Both relations (3.2) correspond to a stratified fluid-solid medium with interfaces $z = \text{const}$.

In accordance with the first Eq. (3.2) and formulas (1.4) and (2.4), the relations

$$\rho'_{11} = (1 - \varepsilon)\rho_s, \quad \rho'_{12} = 0, \quad \rho'_{22} = \varepsilon\rho_f, \quad \delta' = \rho_s\rho_f\varepsilon(1 - \varepsilon) \quad (3.12)$$

are satisfied. If we substitute the second equality (3.2) into the third and sixth equations of a continuous medium (1.3), then the latter equations become indeterminate because ρ''_{11} , ρ''_{12} , and ρ''_{22} tend to infinity. To remove this indeterminacy, we write the right-hand sides of the equations in question:

$$\begin{aligned} \rho_s(1 - \varepsilon) \frac{\partial^2 u_z^{(2)}}{\partial t^2} + \varepsilon\rho_f(\alpha_2 - 1) \left(\frac{\partial^2 u_z^{(2)}}{\partial t^2} - \frac{\partial^2 u_z^{(1)}}{\partial t^2} \right), \\ \rho_f\varepsilon \frac{\partial^2 u_z^{(1)}}{\partial t^2} - \varepsilon\rho_f(\alpha_2 - 1) \left(\frac{\partial^2 u_z^{(2)}}{\partial t^2} - \frac{\partial^2 u_z^{(1)}}{\partial t^2} \right). \end{aligned} \quad (3.13)$$

In order to keep expressions (3.13) finite as $\alpha_2 \rightarrow \infty$, it is necessary to assume that along the z -axis the displacements in both phases coincide, i.e.,

$$u_z^{(1)} = u_z^{(2)} = u_z. \quad (3.14)$$

If, under assumption (3.14), we take into account relations (3.12) and pass, in accordance with formulas (1.1), to the stresses $\tau_{xz}^{(2)}$, $\tau_{zz}^{(2)} = \tau^{(1)}$, then the equations of a continuous medium (1.3) in the two-dimensional case take the form

$$\begin{aligned} \frac{\partial \tau_{xz}^{(2)}}{\partial x} = \rho_s \frac{\partial^2 u_x^{(2)}}{\partial t^2}, \quad \frac{\partial \tau_{zz}^{(2)}}{\partial z} = \rho_s \frac{\partial^2 u_z^{(2)}}{\partial t^2}, \\ \frac{\partial \tau^{(1)}}{\partial x} = \rho_f \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \quad \frac{\partial \tau^{(1)}}{\partial z} = \rho_f \frac{\partial^2 u_z^{(1)}}{\partial t^2}. \end{aligned} \quad (3.15)$$

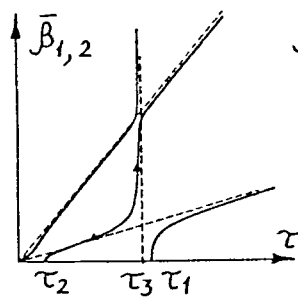


FIG. 9

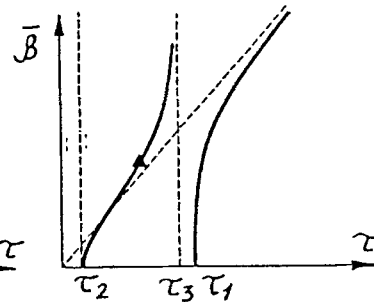


FIG. 10

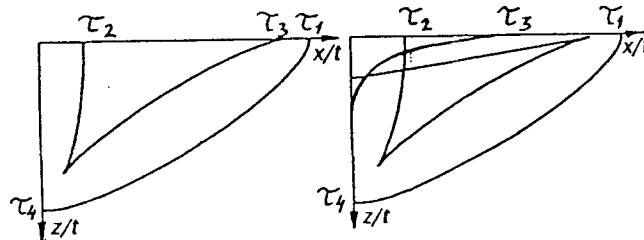


FIG. 11

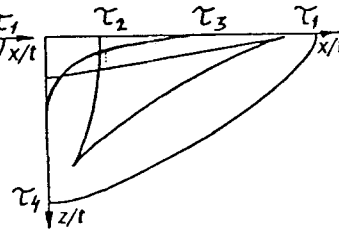


FIG. 12

A similar indeterminacy arises in the expression (1.5) for the density of kinetic energy. If we remove this indeterminacy, use relation (3.12), and pass to the two-dimensional case, then formulas (1.5) are replaced by the equalities

$$\begin{aligned} W_k &= \frac{1}{2} \left[(1 - \varepsilon) \rho_s \left(\frac{\partial u_x^{(2)}}{\partial t} \right)^2 + \varepsilon \rho_f \left(\frac{\partial u_x^{(1)}}{\partial t} \right)^2 + \bar{\rho} \left(\frac{\partial u_z^{(2)}}{\partial t} \right)^2 \right], \\ W_p &= \frac{1}{2} \left\{ P \left(\frac{\partial u_x^{(2)}}{\partial x} \right)^2 + \frac{2F}{1 - \varepsilon} \frac{\partial u_x^{(2)}}{\partial x} \left[(1 - \varepsilon) \frac{\partial u_z^{(2)}}{\partial z} + \varepsilon \frac{\partial u_z^{(1)}}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right] + \right. \\ &\quad \left. + \frac{C}{(1 - \varepsilon)^2} \left[(1 - \varepsilon) \frac{\partial u_z^{(2)}}{\partial z} + \varepsilon \frac{\partial u_z^{(1)}}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right]^2 \right\}, \end{aligned} \quad (3.16)$$

where $\bar{\rho} = \rho_s(1 - \varepsilon) + \rho_f \varepsilon$ is the averaged density.

In Eqs. (3.5) and (3.15) and also in the expression for the density of potential energy (3.16), the derivatives $\partial u_z^{(1)}/\partial z$ and $\partial u_z^{(2)}/\partial z$ occur only in the sum $\varepsilon \partial u_z^{(1)}/\partial z + (1 - \varepsilon) \partial u_z^{(2)}/\partial z$, which is the derivative

$$\frac{\partial u_z}{\partial z} = \varepsilon \frac{\partial u_z^{(1)}}{\partial z} + (1 - \varepsilon) \frac{\partial u_z^{(2)}}{\partial z} \quad (3.17)$$

averaged upon the total volume. A similar derivative may be introduced for the stresses via the formula

$$\frac{\partial \tau_{zz}}{\partial z} = \varepsilon \frac{\partial \tau_{zz}^{(1)}}{\partial z} + (1 - \varepsilon) \frac{\partial \tau_{zz}^{(2)}}{\partial z}. \quad (3.18)$$

If we take into account formulas (3.17) and (3.18) and use the stresses $\tau_{xx}^{(2)}, \tau_{zz}^{(2)} = \tau^{(1)}$, then Eqs. (3.5) and (3.15) are replaced by the relations

$$\begin{aligned} \tau_{xx}^{(2)} &= \frac{P}{1 - \varepsilon} \frac{\partial u_x^{(2)}}{\partial x} + \frac{F}{(1 - \varepsilon)^2} \left(\frac{\partial u_z}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right), \\ \tau_{zz}^{(2)} &= \tau^{(1)} = \frac{F}{1 - \varepsilon} \frac{\partial u_x^{(2)}}{\partial x} + \frac{C}{(1 - \varepsilon)^2} \left(\frac{\partial u_z}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right), \\ \frac{\partial \tau_{xx}^{(2)}}{\partial x} &= \rho_s \frac{\partial^2 u_x^{(2)}}{\partial t^2}, \quad \frac{\partial \tau_{xx}^{(1)}}{\partial x} = \rho_f \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \quad \frac{\partial \tau^{(1)}}{\partial z} = \bar{\rho} \frac{\partial^2 u_z}{\partial t^2}, \end{aligned} \quad (3.19)$$

whereas the energy densities take the form

$$\begin{aligned} W_k &= \frac{1}{2} \left[(1 - \varepsilon) \rho_s \left(\frac{\partial u_x^{(2)}}{\partial t} \right)^2 + \varepsilon \rho_f \left(\frac{\partial u_x^{(1)}}{\partial t} \right)^2 + \bar{\rho} \left(\frac{\partial u_z^{(2)}}{\partial t} \right)^2 \right], \\ W_p &= \frac{1}{2} \left[P \left(\frac{\partial u_x^{(2)}}{\partial x} \right)^2 + \frac{2F}{1 - \varepsilon} \frac{\partial u_x^{(2)}}{\partial x} \left(\frac{\partial u_z}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right) + \frac{C}{(1 - \varepsilon)^2} \left(\frac{\partial u_z}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right)^2 \right]. \end{aligned} \quad (3.20)$$

In accordance with relations (3.6), (3.12), (1.4), (1.6), and (2.4), the function $\bar{\beta}(\tau)$, determining the fronts, is expressed by the formula

$$\bar{\beta}^2 = \frac{\bar{\rho} \left\{ \rho_s \rho_f (1 - \varepsilon)^3 \tau^4 - \tau^2 (1 - \varepsilon) \left[P \rho_f (1 - \varepsilon) + C \rho_s \varepsilon \right] + \varepsilon (PC - F^2) \right\}}{\rho_f \left[C \rho_s (1 - \varepsilon) \tau^2 - PC + F^2 \right]}. \quad (3.21)$$

Since expressions (3.6) and (3.21) are of similar structure, wave fronts in media 2 and 3 differ only in their quantitative characteristics. As previously, fronts in medium 3 correspond to those represented in Fig. 11, but the velocities of waves along the axes are now given by the formulas

$$\begin{aligned}\tau_{1,2}^2 &= \frac{P\rho_f(1-\varepsilon) + C\rho_s\varepsilon \pm \sqrt{\left[P\rho_f(1-\varepsilon) + C\rho_s\varepsilon\right]^2 - 4\rho_s\rho_f\varepsilon(1-\varepsilon)(PC - F^2)}}{2\rho_s\rho_f(1-\varepsilon)^2}, \\ \tau_3^2 &= \frac{PC - F^2}{C\rho_s(1-\varepsilon)}, \quad \tau_4^2 = \frac{C}{\bar{\rho}(1-\varepsilon)^2}.\end{aligned}\quad (3.22)$$

§4. THE EFFECTIVE MODEL OF A LAYERED PERIODIC ELASTIC-FLUID MEDIUM

According to [5], the effective model of a layered periodic elastic-fluid medium is described by the equations of Hooke's law

$$\begin{aligned}\tau_{xx}^{(2)} &= \lambda_0 b \left[\frac{\partial u_z}{\partial z} + \varepsilon \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} \right) \right] + [a + (1-\varepsilon)\lambda_0 b^2] \frac{\partial u_x^{(2)}}{\partial x} + [2\mu b + (1-\varepsilon)\lambda_0 b^2] \frac{\partial u_y^{(2)}}{\partial y}, \\ \tau_{yy}^{(2)} &= \lambda_0 b \left[\frac{\partial u_z}{\partial z} + \varepsilon \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} \right) \right] + [2\mu b + (1-\varepsilon)\lambda_0 b^2] \frac{\partial u_x^{(2)}}{\partial x} + [a + (1-\varepsilon)\lambda_0 b^2] \frac{\partial u_y^{(2)}}{\partial y}, \\ \tau_{zz}^{(2)} &= \tau^{(1)} = \lambda_0 \left[\frac{\partial u_z}{\partial z} + \varepsilon \left(\frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} \right) \right] + (1-\varepsilon)\lambda_0 b \left(\frac{\partial u_x^{(2)}}{\partial x} + \frac{\partial u_y^{(2)}}{\partial y} \right), \\ \tau_{xy}^{(2)} &= \mu \left(\frac{\partial u_x^{(2)}}{\partial y} + \frac{\partial u_y^{(2)}}{\partial x} \right)\end{aligned}\quad (4.1)$$

and the equations of a continuous medium

$$\begin{aligned}\frac{\partial \tau_{xx}^{(2)}}{\partial x} + \frac{\partial \tau_{xy}^{(2)}}{\partial y} &= \rho_s \frac{\partial u_x^{(2)}}{\partial t^2}, \quad \frac{\partial \tau_{xy}^{(2)}}{\partial x} + \frac{\partial \tau_{yy}^{(2)}}{\partial y} = \rho_s \frac{\partial u_y^{(2)}}{\partial t^2}, \\ \frac{\partial \tau^{(1)}}{\partial x} &= \rho_f \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \quad \frac{\partial \tau^{(1)}}{\partial y} = \rho_f \frac{\partial^2 u_y^{(1)}}{\partial t^2}, \quad \frac{\partial \tau_{zz}^{(2)}}{\partial z} = \bar{\rho} \frac{\partial^2 u_z}{\partial t^2},\end{aligned}\quad (4.2)$$

where we use the notation

$$a = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu}, \quad b = \frac{\lambda}{\lambda + 2\mu}, \quad \lambda_0 = \left[\frac{\varepsilon}{\lambda_1} + \frac{1-\varepsilon}{\lambda + 2\mu} \right]^{-1}.\quad (4.3)$$

Here, λ and μ are the Lamé coefficients of the material of the elastic phase, whereas λ_1 is the Lamé coefficient of the material of the fluid phase.

In the two-dimensional case, where the displacement field is independent of the y -coordinate and does not involve the components $u_y^{(1)}$ and $u_y^{(2)}$, Eqs. (4.1) and (4.2) are replaced by

$$\begin{aligned}\tau_{xx}^{(2)} &= [a + (1-\varepsilon)\lambda_0 b^2] \frac{\partial u_x^{(2)}}{\partial x} + \lambda_0 b \left(\frac{\partial u_z}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right), \\ \tau_{zz}^{(2)} &= \tau^{(1)} = \lambda_0 b (1-\varepsilon) \frac{\partial u_x^{(2)}}{\partial x} + \lambda_0 \left(\frac{\partial u_z}{\partial z} + \varepsilon \frac{\partial u_x^{(1)}}{\partial x} \right), \\ \frac{\partial \tau_{xx}^{(2)}}{\partial x} &= \rho_s \frac{\partial^2 u_x^{(2)}}{\partial t^2}, \quad \frac{\partial \tau^{(1)}}{\partial x} = \rho_f \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \quad \frac{\partial \tau_{zz}^{(2)}}{\partial z} = \bar{\rho} \frac{\partial^2 u_z}{\partial t^2}.\end{aligned}\quad (4.4)$$

A comparison of Eqs. (4.4) and (3.19) shows that they coincide if

$$P = (1 - \varepsilon)[a + (1 - \varepsilon)\lambda_0 b^2], \quad F = \lambda_0 b(1 - \varepsilon)^2, \quad C = \lambda_0(1 - \varepsilon)^2. \quad (4.5)$$

From relations (4.5) and (3.4) we obtain the following expressions for the remaining three coefficients of Hooke's law:

$$M = \lambda_0 b \varepsilon(1 - \varepsilon), \quad Q = \lambda_0 \varepsilon(1 - \varepsilon), \quad R = \lambda_0 \varepsilon^2. \quad (4.6)$$

Thus, in the two-dimensional case, under conditions (2.1), (3.2), (4.5), and (4.6), the $P_2 - P_1 - SV$ Biot model in the case where the densities ρ_s , ρ_f and porosity ε are the same, converts into the effective model of a layered elastic-fluid medium.

The fronts of waves excited by a point source in the effective model of a layered elastic-fluid medium are still of the form shown in Fig. 11, but, in accordance with (3.22) and (4.3), the velocities of waves along the axes are now expressed by the equalities

$$\tau_{1,2}^2 = \left\{ \pm \sqrt{[\rho_f a + \rho_f \lambda_0 b^2(1 - \varepsilon) + \lambda_0 \rho_s \varepsilon]^2 - 4\rho_s \rho_f a \varepsilon} + \rho_f [a + \lambda_0 b^2(1 - \varepsilon)] + \lambda_0 \rho_s \varepsilon \right\} / (2\rho_s \rho_f), \quad (4.7)$$

$$\tau_3^2 = a/\rho_s, \quad \tau_4^2 = \lambda_0/\rho.$$

Formulas (4.7) and also the pattern of wave fronts (see Fig. 11) in the model considered were derived in [8, 10, 12, 13]. In [8, 10], the derivation was based on the equations of the effective model, whereas in [12, 13] only the low-frequency limit passage was used because there the equations mentioned were not obtained. The expression for the velocity t_3 is missing in the latter papers. The effective model was experimentally verified in [13].

In order to establish the interrelationship between the two models in the three-dimensional case, it is sufficient to compare additionally the propagation of SH waves in both models. In the Biot model, we consider the wave fields that contain only the components $u_y^{(1)}$ and $u_y^{(2)}$ of displacements, which are independent of the y -coordinate. Based on formulas (1.2) and (1.3), we obtain the relations

$$\begin{aligned} \sigma_{xy}^{(2)} &= N \frac{\partial u_y^{(2)}}{\partial x}, \quad \sigma_{yz}^{(2)} = L \frac{\partial u_y^{(2)}}{\partial z}, \\ \frac{\partial \sigma_{xy}^{(2)}}{\partial x} + \frac{\partial \sigma_{yz}^{(2)}}{\partial z} &= \rho'_{11} \frac{\partial u_y^{(2)}}{\partial t^2} + \rho'_{12} \frac{\partial u_y^{(1)}}{\partial t^2}, \\ \rho'_{12} \frac{\partial u_y^{(2)}}{\partial t^2} + \rho'_{22} \frac{\partial u_y^{(1)}}{\partial t^2} &= 0, \end{aligned} \quad (4.8)$$

which imply the equation for the displacement $u_y^{(2)}$

$$N \frac{\partial^2 u_y^{(2)}}{\partial x^2} + L \frac{\partial^2 u_y^{(2)}}{\partial z^2} = \frac{\rho'_{11} \rho'_{22} - \rho'^2_{12}}{\rho'^2_{22}} \frac{\partial^2 u_y^{(2)}}{\partial t^2}. \quad (4.9)$$

In the case of a point source, Eq. (4.9) leads to an elliptic wave front with velocities

$$\sqrt{\frac{N \rho'_{22}}{\rho'_{11} \rho'_{22} - \rho'^2_{12}}}, \quad \sqrt{\frac{L \rho'_{22}}{\rho'_{11} \rho'_{22} - \rho'^2_{12}}}$$

along the x - and z -axes, respectively. When we pass to medium 1, where $L = 0$, the ellipse transforms into a segment along the x -axis. In medium 3, where (3.12) holds, Eq. (4.8) is replaced by the equality

$$N \frac{\partial^2 u_y^{(2)}}{\partial x^2} = \rho'_{11} \frac{\partial^2 u_y^{(2)}}{\partial t^2}, \quad (4.10)$$

and the velocity of motion of the endpoint of this segment is given by

$$V = \sqrt{\frac{N}{(1-\varepsilon)\rho_s}}. \quad (4.11)$$

In the case of the effective model of a layered fluid-solid medium, a similar wave field is described by the equations

$$\tau_{xy}^{(2)} = \mu \frac{\partial u_y^{(2)}}{\partial x}, \quad \frac{\partial \tau_{xy}^{(2)}}{\partial x} = \rho_s \frac{\partial^2 u_y^{(2)}}{\partial t^2} \quad (4.12)$$

if we take into account relations (4.1) and (4.2).

Equations (4.13) imply the relation

$$\mu \frac{\partial^2 u_y^{(2)}}{\partial x^2} = \rho_s \frac{\partial^2 u_y^{(2)}}{\partial t^2} \quad (4.13)$$

and the expression

$$V = \sqrt{\frac{\mu}{\rho_s}} \quad (4.14)$$

for the velocity of wave propagation along the x -axis. A comparison of Eqs. (4.10) and (4.13) and of formulas (4.11) and (4.14) yields the relation

$$N = \mu(1-\varepsilon). \quad (4.15)$$

Thus, the effective model of a layered periodic fluid-solid medium is a particular case of the Biot model of a porous fluid-saturated medium, provided that

$$\begin{aligned} P &= (1-\varepsilon)[a + (1-\varepsilon)\lambda_0 b^2], \quad A = (1-\varepsilon)b[2\mu + (1-\varepsilon)\lambda_0 b], \\ F &= \lambda_0 b(1-\varepsilon)^2, \quad M = \lambda_0 b\varepsilon(1-\varepsilon), \quad C = \lambda_0(1-\varepsilon)^2, \quad Q = \lambda_0\varepsilon(1-\varepsilon), \\ R &= \lambda_0\varepsilon^2, \quad L = 0, \quad N = \mu(1-\varepsilon), \quad \alpha_1 = 1, \quad \alpha_2 = \infty, \end{aligned} \quad (4.16)$$

where we use notation (4.3), and the densities ρ_s , ρ_f and porosity ε are the same for both models.

In conclusion, we note that both models were generalized to cases of more than two phases. In [8, 9] multiphase models of layered elastic-fluid media with a slipping contact on the interfaces were derived and investigated. The two-phase effective model of a stratified solid-fluid medium is a particular case of the latter models. The Biot model was extended to the case of three phases. In [14], such an extension was provided for the model with two elastic phases and one fluid phase, whereas in [15] the model with one solid, one liquid, and one gaseous phase was investigated.

Without any doubts, the multiphase models mentioned are interrelated in a similar way.

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