

EFFECTIVE MODELS OF STRATIFIED MEDIA CONTAINING POROUS BIOT LAYERS

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UDC 550.310

Periodic stratified media in which either two porous Biot layer, or an elastic and a porous layers, or a fluid and a porous layer alternate are considered. The effective models of these media are constructed and investigated. In the case of alternating porous layers, the effective model is a generalized transversely isotropic Biot medium. In this medium, the density of the fluid phase and the mean density acquire tensor character. It is shown that the effective model of a porous-fluid medium is, on the one hand, a generalized transversely isotropic Biot medium of special type and, on the other hand, a generalization of the effective model of a stratified elastic-fluid medium. Bibliography: 14 titles.

At present, due to advancements in seismic methods of exploring, in the theory of seismic wave propagation, and in computer-aided methods, active interest in wave propagation in porous and fractured media is being expressed. The investigation of wave processes in such media is typically carried out by using the two-phase Biot model [1, 2]. Due to the fast development of the theory of these processes, the study of rather complicated stratified structures containing porous as well as elastic or fluid layers has become possible. In the present paper, we construct and investigate the effective models for stratified periodic media with two kinds of alternating layers, which are either two Biot layers, or a Biot layer and an elastic layer, or a Biot layer and a fluid layer.

Although such periodic structures rarely occur in nature, the study of them is of interest in considering waves in porous media with heterogeneities and inclusions of different kinds. Since for the media indicated it is in general impossible to construct wave fields, the investigation of these media can be carried out only by using effective models of them. These effective models can be constructed in the cases where comparatively simple expressions for wave fields exist. The case of layered periodic structures is one such case.

The method used in deriving the equations of effective models was suggested in 1979 (see [3]). It is referred to as the method of matrix averaging because it uses exact solutions represented in matrix form. By using the method of matrix averaging, effective models for elastic, fluid, and elastic-fluid layered media with account for anisotropy [4, 5], viscosity [6, 7], relaxation kernels [6], and complicated linear contacts on interfaces [8] have already been constructed. In deriving new effective models, the method of matrix averaging was elaborated and improved, which led to the development of a simplified version of this method [9].

In accordance with this version, the equations of an effective model are obtained as a result of simple formal operations, which are easy to justify by using the exact solutions of the equations that describe a given periodic medium. In this paper, the simplified version of the method of matrix averaging is used.

Due to averaging, in some cases effective models prove to be more complicated than the Biot medium. Such models are called generalized transversely isotropic Biot media. The effective models differ from Biot's media only in the equations of a continuous medium, which become more complicated due to the anisotropy of the fluid phase. To conclude this introduction, we mention that layered periodic Biot media were already considered in [10–12], where averaging was applied to some of the parameters but no effective model together with the differential equations describing it was provided.

1. THE BIOT MEDIUM

Let a layered periodic medium $0 \leq z \leq H$ containing n periods of thickness $h = H/n$ be given. Every period consists of two homogeneous layers 1 and 2 of thicknesses $h_1 = \theta_1 h$ and $h_2 = \theta_2 h$, respectively

(Fig. 1). In the first case under consideration, both layers are described by the equations

$$\begin{aligned}\tau_{xx} &= P \frac{\partial u_x}{\partial x} + F \frac{\partial u_z}{\partial z} + M \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right), \\ \tau_{zz} &= F \frac{\partial u_x}{\partial x} + C \frac{\partial u_z}{\partial z} + Q \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right), \\ -p &= M \frac{\partial u_x}{\partial x} + Q \frac{\partial u_z}{\partial z} + R \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right),\end{aligned}\tag{1.1}$$

$$\begin{aligned}\tau_{xz} &= L \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right); \\ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} &= \bar{\rho} \frac{\partial^2 u_x}{\partial t^2} + \rho_f \frac{\partial^2 w_x}{\partial t^2}, \\ -\frac{\partial p}{\partial x} &= \rho_f \frac{\partial^2 u_x}{\partial t^2} + m_x \frac{\partial^2 w_x}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} &= \bar{\rho} \frac{\partial^2 u_z}{\partial t^2} + \rho_f \frac{\partial^2 w_z}{\partial t^2}, \\ -\frac{\partial p}{\partial z} &= \rho_f \frac{\partial^2 u_z}{\partial t^2} + m_z \frac{\partial^2 w_z}{\partial t^2}\end{aligned}\tag{1.2}$$

of the transversely isotropic Biot medium. Equalities (1.1) express Hooke's law, whereas relations (1.2) are the equations of a continuous medium. Equations (1.1) and (1.2) are characterized by 11 parameters: 7 moduli and 4 quantities with dimensionality of density. The parameter ρ_f is the density of the fluid phase, and the remaining three parameters are given by the formulas

$$\begin{aligned}\bar{\rho} &= \rho_f f + \rho_s(1 - f), \\ m_i &= \rho_f \alpha_i / f \quad (i = x, z),\end{aligned}\tag{1.3}$$

where ρ_s is the density of the solid phase, and α_x , α_z are the tortuosities of pores along the x and z axes ($\alpha_i \geq 1$). The Biot medium is also defined by the additional porosity parameter f . In order to pass to the isotropic Biot medium, it is sufficient to take into account the equalities $P = C$, $M = Q$, and $m_x = m_z$.

The stresses τ_{xx} , τ_{zz} , and τ_{xz} are the total stresses. In terms of the stresses $\sigma_{xx}^{(2)}$, $\sigma_{zz}^{(2)}$, $\sigma_{xz}^{(2)}$ of the solid phase and of the pressure p in the fluid phase, they are expressed as

$$\tau_{xx} = \sigma_{xx}^{(2)} - pf, \quad \tau_{zz} = \sigma_{zz}^{(2)} - pf, \quad \tau_{xz} = \sigma_{xz}^{(2)}.\tag{1.4}$$

The displacements u_x , u_z , w_x , and w_z are given by the relations

$$\begin{aligned}u_x &= u_x^{(2)}, \quad u_z = u_z^{(2)}, \\ w_x &= f(u_x^{(1)} - u_x^{(2)}), \quad w_z = f(u_z^{(1)} - u_z^{(2)}),\end{aligned}\tag{1.5}$$

where $u_x^{(1)}$, $u_z^{(1)}$ and $u_x^{(2)}$, $u_z^{(2)}$ are the displacements in the fluid and elastic phases, respectively.

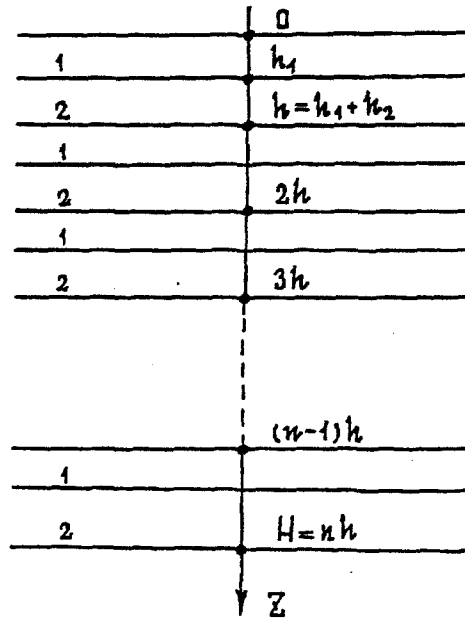


FIG. 1

The presence of two phases in the Biot medium is related to the fact that, in a small neighborhood of an arbitrary point of this medium, both fluid and solid media are present. The displacements $u_x^{(1)}$, $u_z^{(1)}$ and the pressure p are averaged over the fluid phase localized in a small neighborhood of the point under consideration. Similarly, the displacements $u_x^{(2)}$ and $u_z^{(2)}$ are obtained as a result of averaging over the elastic medium in a small neighborhood. The stresses $\sigma_{xx}^{(2)}$, $\sigma_{zz}^{(2)}$, and $\sigma_{xz}^{(2)}$ result from averaging over the whole volume of the small neighborhood, and, in this case, the stresses subjected to averaging are assumed to be zero in the fluid phase.

The matrix composed of the coefficients in Hooke's law (1.1) is symmetric and positive definite. The symmetry of this matrix is due to averaging the displacements and stresses as described above. The positive definiteness of the matrix is determined by the positive definiteness of the density of potential energy. For the density of kinetic energy to be positive definite it is necessary that the inequalities

$$\bar{\rho}m_i - \rho_f^2 > 0 \quad (i = x, z) \quad (1.6)$$

be satisfied.

On the interfaces between layers 1 and 2, the conditions

$$\begin{aligned} u_{x1} &= u_{x2}, & u_{z1} &= u_{z2}, & w_{z1} &= w_{z2}, \\ \tau_{zx1} &= \tau_{zx2}, & \tau_{xz1} &= \tau_{xz2}, & p_1 &= p_2 \end{aligned} \quad (1.7)$$

are fulfilled, where the subscripts 1 and 2, added to all the quantities, indicate the number of the medium. Boundary conditions (1.7) are analogous to the conditions of rigid contact between elastic media. In the case of identical Biot media, on interfaces with conditions (1.7) no reflected waves arise. Boundary conditions (1.7) take into account the continuity of the total stresses, of the medium, and of the energy streams through the interfaces.

2. THE EFFECTIVE MODEL OF A PERIODIC MEDIUM WITH POROUS BIOT LAYERS

If the source and conditions on the outer interfaces $z = 0$ and $z = H$ are given, then Eqs. (1.1), (1.2) and conditions (1.7) on the interfaces between layers permit one to obtain the exact expressions for the fields of displacements and stresses in every layer with account for all possible reflections and refractions. In determining the wave field, integral transforms are applied and, in order to derive a compact representation, the matrix method is used. In this interference field, the passage to the limit as $n \rightarrow \infty$ and $h \rightarrow 0$ can be carried out for $H = \text{const}$, $\theta_1 = \text{const}$, and $\theta_2 = \text{const}$. In the limiting wave field, the continuous displacements and stresses and also the averaged discontinuous quantities satisfy new equations, which describe the effective model.

In this paper, in deriving these equations, we apply the simplified approach suggested in [7]. Among the eight quantities occurring in Eqs. (1.1) and (1.2), the six variables u_x , u_z , w_z , τ_{zz} , p , and τ_{xz} are continuous, but w_x and τ_{xx} have jumps on the interfaces. Using Eqs. (1.1) and (1.2), we express the derivatives of the continuous functions with respect to z in terms of these functions and their derivatives with respect to x and t . As a result, we obtain the equations

$$\begin{aligned} \frac{\partial u_x}{\partial z} &= -\frac{\partial u_z}{\partial x} + \frac{1}{L} \tau_{xz}, \\ \frac{\partial u_z}{\partial z} &= -\frac{FR - MQ}{CR - Q^2} \frac{\partial u_x}{\partial x} + \frac{R}{CR - Q^2} \tau_{zz} + \frac{Q}{CR - Q^2} p, \\ \frac{\partial^3 w_z}{\partial z \partial t^2} &= \left(\frac{\rho_f}{m_x} + \frac{FQ - CM}{CR - Q^2} \right) \frac{\partial^3 u_x}{\partial x \partial t^2} - \frac{Q}{CR - Q^2} \frac{\partial^2 \tau_{zz}}{\partial t^2} + \frac{1}{m_x} \frac{\partial^2 p}{\partial x^2} - \frac{C}{CR - Q^2} \frac{\partial^2 p}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial z} &= \frac{m_x \bar{\rho} - \rho_f^2}{m_x} \frac{\partial^2 u_x}{\partial t^2} - \frac{\Delta}{CR - Q^2} \frac{\partial^2 u_x}{\partial x^2} - \frac{FR - MQ}{CR - Q^2} \frac{\partial \tau_{zz}}{\partial x} - \left(\frac{\rho_f}{m_x} + \frac{FQ - CM}{CR - Q^2} \right) \frac{\partial p}{\partial x}, \\ \frac{\partial \tau_{zz}}{\partial z} &= \bar{\rho} \frac{\partial^2 u_z}{\partial t^2} + \rho_f \frac{\partial^2 w_z}{\partial t^2} - \frac{\partial \tau_{xz}}{\partial x}, \\ -\frac{\partial p}{\partial z} &= \rho_f \frac{\partial^2 u_z}{\partial t^2} + m_z \frac{\partial^2 w_z}{\partial t^2}. \end{aligned} \quad (2.1)$$

The quantity Δ , involved in the fourth equation in (2.1), is the determinant of the matrix formed by the coefficients in Hooke's law (1.1) and can be represented in the form

$$\Delta = PCR + 2FQM - CM^2 - RF^2 - PQ^2. \quad (2.2)$$

Equations (2.1) must be supplemented with two additional equations

$$\begin{aligned} \frac{\partial^2 w_x}{\partial t^2} &= -\frac{\rho_f}{m_x} \frac{\partial^2 u_x}{\partial t^2} - \frac{1}{m_x} \frac{\partial p}{\partial x}, \\ \tau_{xx} &= \frac{\Delta}{CR - Q^2} \frac{\partial u_x}{\partial x} + \frac{FR - MQ}{CR - Q^2} \tau_{zz} + \frac{FQ - CM}{CR - Q^2} p, \end{aligned} \quad (2.3)$$

expressing the discontinuous functions w_x and τ_{xx} in terms of the continuous variables. The averaging of Eqs. (2.1) reduces to the introduction of the following averaged derivatives:

$$\begin{aligned} \frac{\partial u_x}{\partial z} &= \frac{\partial u_{x1}}{\partial z} \theta_1 + \frac{\partial u_{x2}}{\partial z} \theta_2, & \frac{\partial u_z}{\partial z} &= \frac{\partial u_{z1}}{\partial z} \theta_1 + \frac{\partial u_{z2}}{\partial z} \theta_2, \\ \frac{\partial w_z}{\partial z} &= \frac{\partial w_{z1}}{\partial z} \theta_1 + \frac{\partial w_{z2}}{\partial z} \theta_2, & \frac{\partial p}{\partial z} &= \frac{\partial p_1}{\partial z} \theta_1 + \frac{\partial p_2}{\partial z} \theta_2, \\ \frac{\partial \tau_{xz}}{\partial z} &= \frac{\partial \tau_{xz1}}{\partial z} \theta_1 + \frac{\partial \tau_{xz2}}{\partial z} \theta_2, & \frac{\partial \tau_{zz}}{\partial z} &= \frac{\partial \tau_{zz1}}{\partial z} \theta_1 + \frac{\partial \tau_{zz2}}{\partial z} \theta_2. \end{aligned} \quad (2.4)$$

In equalities (2.4), the displacements and stresses related to a fixed layer have an additional subscript indicating the type of the layer. Note that the averaged derivatives have no subscripts. The averaging

based on formulas (2.4) can be strictly justified by carrying out the above-indicated passage to the limit and is consistent with the definition of a derivative. The discontinuous quantities w_x and τ_{xx} are averaged in accordance with the formulas

$$w_x = w_{x1}\theta_1 + w_{x2}\theta_2, \quad \tau_{xx} = \tau_{xx1}\theta_1 + \tau_{xx2}\theta_2. \quad (2.5)$$

These formulas cannot be obtained by passing to the limit. However, since the wave field in the effective model depends on the variables w_{x1} , w_{x2} , τ_{xx1} , and τ_{xx2} , it is useful to express formulas (2.5) in terms of w_x and τ_{xx} only. In addition, the functions w_x and τ_{xx} introduced via (1.5) can be regarded as the mean relative displacement and the stress distributed over the total period. In the effective model, these variables can also be expressed by relations analogous to the third formula in (1.5) and the first equality in (1.4). A similar averaging was used in deriving the equations of an effective model in the case of elastic and fluid stratified periodic systems [3, 7, 13].

Now we focus on the derivation of the equations of an effective model. To this end, we substitute the expressions (2.1) and (2.3) written for the first and second media into the right-hand sides of relations (2.4) and (2.5) and introduce the notation

$$\begin{aligned} A_1 &= \sum_{i=1}^2 \frac{F_i R_i - M_i Q_i}{C_i R_i - Q_i^2} \theta_i, & A_2 &= \sum_{i=1}^2 \frac{F_i Q_i - C_i M_i}{C_i R_i - Q_i^2} \theta_i, \\ A_3 &= \sum_{i=1}^2 \frac{\Delta_i \theta_i}{C_i R_i - Q_i^2}, & A_4 &= \sum_{i=1}^2 \frac{Q_i \theta_i}{C_i R_i - Q_i^2}, \\ A_5 &= \sum_{i=1}^2 \frac{C_i \theta_i}{C_i R_i - Q_i^2}, & A_6 &= \sum_{i=1}^2 \frac{R_i \theta_i}{C_i R_i - Q_i^2}, & A_7 &= \sum_{i=1}^2 \frac{\theta_i}{L_i}, \\ B_1 &= \sum_{i=1}^2 \frac{(m_{xi} \bar{\rho}_i - \rho_{fi}^2) \theta_i}{m_{xi}}, & B_2 &= \sum_{i=1}^2 \frac{\rho_{fi} \theta_i}{m_{xi}}, & B_3 &= \sum_{i=1}^2 \frac{\theta_i}{m_{xi}}, \\ B_4 &= \sum_{i=1}^2 \bar{\rho}_i \theta_i, & B_5 &= \sum_{i=1}^2 \rho_{fi} \theta_i, & B_6 &= \sum_{i=1}^2 m_{xi} \theta_i. \end{aligned} \quad (2.6)$$

As a result we obtain the equations

$$\begin{aligned} \frac{\partial u_x}{\partial z} &= -\frac{\partial u_z}{\partial x} + A_7 \tau_{xz}, \\ \frac{\partial u_z}{\partial z} &= -A_1 \frac{\partial u_x}{\partial x} + A_6 \tau_{zz} + A_4 p, \\ \frac{\partial^3 w_z}{\partial z \partial t^2} &= (A_2 + B_2) \frac{\partial^3 u_x}{\partial x \partial t^2} - A_4 \frac{\partial^2 \tau_{zz}}{\partial t^2} + B_3 \frac{\partial^2 p}{\partial x^2} - A_5 \frac{\partial^2 p}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial z} &= B_1 \frac{\partial^2 u_x}{\partial t^2} - A_3 \frac{\partial^2 u_x}{\partial x^2} - A_1 \frac{\partial \tau_{zz}}{\partial x} - (A_2 + B_2) \frac{\partial p}{\partial x}, \\ \frac{\partial \tau_{zz}}{\partial z} &= B_4 \frac{\partial^2 u_z}{\partial t^2} + B_5 \frac{\partial^2 w_z}{\partial t^2} - \frac{\partial \tau_{xz}}{\partial x}, \\ -\frac{\partial p}{\partial z} &= B_5 \frac{\partial^2 u_z}{\partial t^2} + B_6 \frac{\partial^2 w_z}{\partial t^2}, \\ \frac{\partial^2 w_x}{\partial t^2} &= -B_2 \frac{\partial^2 u_x}{\partial t^2} - B_3 \frac{\partial p}{\partial x}, \\ \tau_{xx} &= A_3 \frac{\partial u_x}{\partial x} + A_1 \tau_{zz} + A_2 p, \end{aligned} \quad (2.7)$$

which describe a homogeneous medium. Relations (2.7) can be transformed to the equations

$$\tau_{xx} = \bar{P} \frac{\partial u_x}{\partial x} + \bar{F} \frac{\partial u_z}{\partial z} + \bar{M} \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right),$$

$$\begin{aligned}
\tau_{zz} &= \bar{F} \frac{\partial u_x}{\partial x} + \bar{C} \frac{\partial u_z}{\partial z} + \bar{Q} \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right), \\
-p &= \bar{M} \frac{\partial u_x}{\partial x} + \bar{Q} \frac{\partial u_z}{\partial z} + \bar{R} \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right), \\
\tau_{xz} &= \bar{L} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right);
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} &= \bar{B}_1 \frac{\partial^2 u_x}{\partial t^2} + \bar{B}_2 \frac{\partial^2 w_x}{\partial t^2}, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} &= B_4 \frac{\partial^2 u_z}{\partial t^2} + B_5 \frac{\partial^2 w_z}{\partial t^2}, \\
-\frac{\partial p}{\partial x} &= \bar{B}_2 \frac{\partial^2 u_x}{\partial t^2} + \bar{B}_3 \frac{\partial^2 w_x}{\partial t^2}, \\
-\frac{\partial p}{\partial z} &= B_5 \frac{\partial^2 u_z}{\partial t^2} + B_6 \frac{\partial^2 w_z}{\partial t^2},
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
\bar{P} &= \frac{A_3 A_5 A_6 - 2 A_1 A_2 A_4 + A_1^2 A_5 + A_2^2 A_6 - A_3 A_4^2}{A_5 A_6 - A_4^2}, \\
\bar{F} &= \frac{A_1 A_5 - A_2 A_4}{A_5 A_6 - A_4^2}, \quad \bar{C} = \frac{A_5}{A_5 A_6 - A_4^2}, \quad \bar{M} = \frac{A_1 A_4 - A_2 A_6}{A_5 A_6 - A_4^2}, \\
\bar{Q} &= \frac{A_4}{A_5 A_6 - A_4^2}, \quad \bar{R} = \frac{A_6}{A_5 A_6 - A_4^2}, \quad \bar{L} = \frac{1}{A_7}, \\
\bar{B}_1 &= \frac{B_1 B_3 + B_2^2}{B_3}, \quad \bar{B}_2 = \frac{B_2}{B_3}, \quad \bar{B}_3 = \frac{1}{B_3}.
\end{aligned} \tag{2.10}$$

Before discussing the equations obtained, we demonstrate that the densities of kinetic and potential energies of the wave field described by Eqs. (2.8) and (2.9) are positive-definite forms. To this end, it is necessary to prove that the matrix composed of the coefficients of Hooke's law (2.8) is positive definite and that the inequalities

$$\bar{B}_1 \bar{B}_3 - \bar{B}_2^2 > 0, \quad B_4 B_6 - B_5^2 > 0, \quad \bar{B}_i > 0 \quad (i = 1, 2, 3), \quad B_k > 0 \quad (k = 4, 5, 6) \tag{2.11}$$

are satisfied. In the course of the proof, the positive definiteness of the matrix formed by the coefficients in relations (1.1) and equalities (1.6) must be used. The proof of relations (2.11) is elementary. In order to prove the positive definiteness of the matrix composed of the coefficients of Eqs. (2.8), it is necessary to ascertain that the inequalities

$$\bar{P} > 0, \quad \bar{C} > 0, \quad \bar{R} > 0, \quad A_5 A_6 - A_4^2 > 0, \quad A_3 A_5 A_6 - 2 A_1 A_2 A_4 + A_1^2 A_5 + A_2^2 A_6 - A_3 A_4^2 > 0 \tag{2.12}$$

are valid. The proof of inequalities (2.12) is based on the inequalities

$$P > 0, \quad C > 0, \quad R > 0, \quad CR - Q^2 > 0, \quad \Delta > 0, \tag{2.13}$$

following from the positive definiteness of the coefficient matrix of Hooke's law (1.1). Thus, from the energy point of view, the wave field in the effective model obtained can exist.

The model described by Eqs. (2.8) and (2.9) is a generalized transversely isotropic Biot medium. This conclusion is based on the similarity between Eqs. (2.8), (2.9) and Eqs. (1.1), (1.2), and on the identical physical meaning of the corresponding variables entering these equations. This generalization of the Biot

model concerns only the equations of a continuous medium, which are of the form

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} &= \bar{\rho}_x \frac{\partial^2 u_x}{\partial t^2} + \rho_{fx} \frac{\partial^2 w_x}{\partial t^2}, \\
-\frac{\partial p}{\partial x} &= \rho_{fx} \frac{\partial^2 u_x}{\partial t^2} + m_x \frac{\partial^2 w_x}{\partial t^2}, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} &= \bar{\rho}_z \frac{\partial^2 u_z}{\partial t^2} + \rho_{fz} \frac{\partial^2 w_z}{\partial t^2}, \\
-\frac{\partial p}{\partial z} &= \rho_{fz} \frac{\partial^2 u_z}{\partial t^2} + m_z \frac{\partial^2 w_z}{\partial t^2}
\end{aligned} \tag{2.14}$$

and coincide with Eqs. (2.9) up to coefficients. Equations (2.9) as well as (2.14) are determined by six parameters, whereas Eqs. (1.2) involve only four parameters. The appearance of two additional parameters is related to the fact that the densities are averaged differently along the x and z axes, and thus the density of the fluid phase and the mean density both acquire tensor character. The difference in the definitions of the densities was pointed out in [3] in deriving the effective model of a fluid layered medium. If we average a periodic medium the period of which consists of two generalized transversely isotropic Biot layers (Fig. 1), then the effective medium will also be described by the equations of a continuous medium (2.14). The expressions for the coefficients of these equations in terms of the corresponding coefficients for each of the layers are as follows:

$$\bar{\rho}_z = \bar{\rho}_{z1}\theta_1 + \bar{\rho}_{z2}\theta_2, \quad \rho_{fz} = \rho_{fz1}\theta_1 + \rho_{fz2}\theta_2, \quad m_z = m_{z1}\theta_1 + m_{z2}\theta_2,$$

$$\begin{aligned}
\bar{\rho}_x &= \bar{\rho}_{x1}\theta_1 + \bar{\rho}_{x2}\theta_2 - \frac{\theta_1\theta_2(\rho_{fx1} - \rho_{fx2})^2}{m_{x2}\theta_1 + m_{x1}\theta_2}, \\
\rho_{fx} &= \frac{\rho_{fx1}m_{x2}\theta_1 + \rho_{fx2}m_{x1}\theta_2}{m_{x2}\theta_1 + m_{x1}\theta_2}, \quad m_x = \frac{m_{x1}m_{x2}}{m_{x2}\theta_1 + m_{x1}\theta_2}.
\end{aligned} \tag{2.15}$$

If we write the obvious expression for the porosity in the effective model

$$f = f_1\theta_1 + f_2\theta_2 \tag{2.16}$$

and take into account the second formula in (1.3), then for the components of tortuosity we obtain the expressions

$$\begin{aligned}
\alpha_x &= \frac{\alpha_{x1}\alpha_{x2}(f_1\theta_1 + f_2\theta_2)}{f_1\alpha_{x2}\theta_1 + f_2\alpha_{x1}\theta_2}, \\
\alpha_z &= \frac{(\alpha_{z1}\rho_{fz1}f_2\theta_1 + \alpha_{z2}\rho_{fz2}f_1\theta_2)(f_1\theta_1 + f_2\theta_2)}{f_1f_2(f_1\alpha_{x2}\theta_1 + f_2\alpha_{x1}\theta_2)}.
\end{aligned} \tag{2.17}$$

In view of the inequalities $\alpha_{x1} \geq 1$, $\alpha_{z1} \geq 1$, $\alpha_{x2} \geq 1$, and $\alpha_{z2} \geq 1$, from (2.17) it follows that $\alpha_x \geq 1$ and $\alpha_z \geq 1$.

Equations (2.15)–(2.17) also hold in the case where the period consists of the ordinary transversely isotropic Biot layers. In this case, the relations

$$\bar{\rho}_{xi} = \bar{\rho}_{zi}, \quad \rho_{fxi} = \rho_{fzi} \quad (i = 1, 2) \tag{2.18}$$

are valid. The case where the transversely isotropic (or isotropic) Biot layers have the same densities of the fluid phases ($\rho_{f1} = \rho_{f2}$) is of special interest. In this case, the effective medium coincides with the transversely isotropic Biot medium.

3. THE EFFECTIVE MODEL OF A STRATIFIED PERIODIC MEDIUM CONSISTING OF ALTERNATING ELASTIC AND POROUS LAYERS

Consider a periodic medium (see Fig. 1) in which every period consists of a porous Biot layer 1 and an elastic layer 2. The porous layers are described by Eqs. (1.1) and (1.2), in which the displacements and stresses are provided with an additional subscript 1. The displacements u_{x2} , u_{z2} and stresses t_{xx2} , t_{zz2} , t_{xz2} in the elastic layers satisfy the known equations

$$\begin{aligned} t_{xx2} &= (\lambda + 2\mu) \frac{\partial u_{x2}}{\partial x} + \lambda \frac{\partial u_{z2}}{\partial z}, \\ t_{zz2} &= \lambda \frac{\partial u_{x2}}{\partial x} + (\lambda + 2\mu) \frac{\partial u_{z2}}{\partial z}, \\ t_{xz2} &= \mu \left(\frac{\partial u_{x2}}{\partial z} + \frac{\partial u_{z2}}{\partial x} \right), \\ \frac{\partial t_{xx2}}{\partial x} + \frac{\partial t_{xz2}}{\partial z} &= \rho \frac{\partial^2 u_{x2}}{\partial t^2}, \\ \frac{\partial t_{xz2}}{\partial x} + \frac{\partial t_{zz2}}{\partial z} &= \rho \frac{\partial^2 u_{z2}}{\partial t^2}. \end{aligned} \quad (3.1)$$

On the interfaces between the layers, the conditions

$$\begin{aligned} u_{x1} &= u_{x2}, \quad u_{z1} = u_{z2}, \quad w_{z1} = 0, \\ \tau_{xz1} &= t_{xz2}, \quad \tau_{zz1} = t_{zz2} \end{aligned} \quad (3.2)$$

are fulfilled. These conditions can be viewed as a special case of boundary conditions (1.7), in which the porosity of the second layer is zero. As a consequence, the sixth condition in (1.7) must be rejected because it becomes meaningless. Boundary conditions (3.2) take into account the continuity of the total stresses, of the energy stream, and of the medium.

As in Sec. 2, the derivation of the equations of the effective model is based on the simplified version of the matrix method of averaging. In accordance with this version, the equality

$$w_z = 0, \quad (3.3)$$

valid in the whole region, follows from the third boundary condition in (3.2) because, as we are passing from the medium to the effective model, the interfaces become infinitely close. Under these conditions, the derivative $\partial p / \partial z$, being the derivative of a discontinuous function, loses its meaning. Therefore, the last equations in (1.2) and (2.1), involving the derivative $\partial p / \partial z$, must be dropped.

Upon taking into account relations (3.3) and eliminating one of the equations, the Biot layers that are in contact with the elastic layers are described by seven equations:

$$\begin{aligned} \frac{\partial u_{x1}}{\partial z} &= -\frac{\partial u_{z1}}{\partial z} + \frac{1}{L} \tau_{xz1}, \\ \frac{\partial u_{z1}}{\partial z} &= -\frac{FR - MQ}{CR - Q^2} \frac{\partial u_{x1}}{\partial x} + \frac{R}{CR - Q^2} \tau_{zz1} + \frac{Q}{CR - Q^2} p_1, \\ \frac{\partial \tau_{xz1}}{\partial z} &= \frac{m_x \bar{\rho} - \rho_f^2}{m_x} \frac{\partial^2 u_{x1}}{\partial t^2} - \frac{\Delta}{CR - Q^2} \frac{\partial^2 u_x}{\partial x^2} - \frac{FR - MQ}{CR - Q^2} \frac{\partial \tau_{zz}}{\partial x} - \left(\frac{\rho_f}{m_x} + \frac{FQ - CM}{CR - Q^2} \right) \frac{\partial p}{\partial x}, \\ \frac{\partial \tau_{zz1}}{\partial z} &= \bar{\rho} \frac{\partial^2 u_{z1}}{\partial t^2} + \rho_f \frac{\partial^2 w_{z1}}{\partial t^2} - \frac{\partial \tau_{xz}}{\partial x}, \\ \tau_{xz1} &= \frac{\Delta}{CR - Q^2} \frac{\partial u_{x1}}{\partial x} + \frac{FR - MQ}{CR - Q^2} \tau_{zz1} + \frac{FQ - CM}{CR - Q^2} p_1, \\ \frac{\partial^2 w_{z1}}{\partial t^2} &= -\frac{\rho_f}{m_x} \frac{\partial^2 u_{x1}}{\partial t^2} - \frac{1}{m_x} \frac{\partial p_1}{\partial x}, \\ \frac{1}{m_x} \frac{\partial^2 p_1}{\partial x^2} - \frac{C}{CR - Q^2} \frac{\partial^2 p_1}{\partial t^2} &= \frac{Q}{CR - Q^2} \frac{\partial^2 \tau_{zz1}}{\partial t^2} - \left(\frac{\rho_f}{m_x} + \frac{FQ - CM}{CR - Q^2} \right) \frac{\partial^3 u_{x1}}{\partial x \partial t^2}, \end{aligned} \quad (3.4)$$

which are implied by equalities (2.1) and (2.3). Similar equations

$$\begin{aligned}
\frac{\partial u_{x2}}{\partial z} &= -\frac{\partial u_{z2}}{\partial x} + \frac{1}{\mu} \tau_{xz2}, \\
\frac{\partial u_{z2}}{\partial z} &= -\frac{\lambda}{\lambda + 2\mu} \frac{\partial u_{x2}}{\partial x} + \frac{1}{\lambda + 2\mu} \tau_{zz2}, \\
\frac{\partial \tau_{xz2}}{\partial z} &= \rho \frac{\partial^2 u_{x2}}{\partial t^2} - \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial^2 u_{x2}}{\partial x^2} - \frac{\lambda}{\lambda + 2\mu} \frac{\partial \tau_{zz2}}{\partial x}, \\
\frac{\partial \tau_{zz2}}{\partial z} &= \rho \frac{\partial^2 u_{z2}}{\partial t^2} - \frac{\partial \tau_{xz2}}{\partial x}, \\
\tau_{xx2} &= \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial u_{x2}}{\partial x} + \frac{\lambda}{\lambda + 2\mu} \tau_{zz2}
\end{aligned} \tag{3.5}$$

hold in the case of an elastic medium. Relations (3.4) and (3.5) involve four continuous functions u_x , u_z , τ_{xz} , and τ_{zz} , provided with additional subscripts 1 and 2 in the porous and elastic layers, respectively. The discontinuous function τ_{xx} , which is equal to τ_{xx1} in the porous layers and to τ_{xx2} in the elastic layers, is averaged in accordance with the formula $\tau_{xx} = \theta_1 \tau_{xx1} + \theta_2 \tau_{xx2}$. Instead of the variable $w_{x1} = f_1(u_x^{(1)} - u_x)$, determining the relative displacement in a porous layer, the function $w_x = f_1 \theta_1 (u_x^{(1)} - u_x) = w_{x1} \theta_1$, involving the porosity $f = f_1 \theta_1$ in the whole effective model, is introduced. Since the pressure p_1 has no counterparts in the elastic layers and is defined only in porous media, the expression and physical meaning of this quantity remain unchanged upon passing to the effective model, and we have $p = p_1$.

The right-hand sides of Eqs. (3.4) and (3.5) involve only continuous functions and their derivatives with respect to t and x , with the exception of the pressure p and its derivative $\partial p / \partial x$. If we use the operator

$$\left(\frac{1}{m_x} \frac{\partial^2}{\partial x^2} - \frac{C}{CR - Q^2} \frac{\partial^2}{\partial t^2} \right)^{-1},$$

then the pressure p and its derivative will be eliminated from the right-hand sides of relations (3.4), and we will arrive at a situation similar to that considered in Sec. 2. However, such an operation is not needed because averaging does not affect the pressure p , and the seventh equation in (3.4) remains unchanged. Using this equation, upon averaging we can return to terms involving p and $\partial p / \partial x$.

Applying the approach presented in Sec. 2, we write the five averaged equations

$$\begin{aligned}
\frac{\partial u_x}{\partial z} &= -\frac{\partial u_z}{\partial x} + A_7 \tau_{xz}, \\
\frac{\partial u_z}{\partial z} &= -A_1 \frac{\partial u_x}{\partial x} + A_6 \tau_{zz} + A_4 p, \\
\frac{\partial \tau_{xz}}{\partial z} &= B_1 \frac{\partial^2 u_x}{\partial t^2} - A_3 \frac{\partial^2 u_x}{\partial x^2} - A_1 \frac{\partial \tau_{zz}}{\partial x} - (A_2 + B_2) \frac{\partial p}{\partial x}, \\
\frac{\partial \tau_{zz}}{\partial z} &= B_4 \frac{\partial^2 u_z}{\partial t^2} - \frac{\partial \tau_{xz}}{\partial x}, \\
\tau_{xx} &= A_3 \frac{\partial u_x}{\partial x} + A_1 \tau_{zz} + A_2 p
\end{aligned} \tag{3.6}$$

and the two equations subject to no averaging

$$\begin{aligned}
\frac{\partial^2 w_x}{\partial t^2} &= -B_2 \frac{\partial^2 u_x}{\partial t^2} - B_3 \frac{\partial p}{\partial x}, \\
(A_2 + B_2) \frac{\partial^3 u_x}{\partial x \partial t^2} - A_4 \frac{\partial^2 \tau_{zz}}{\partial t^2} + B_3 \frac{\partial^2 p}{\partial x^2} - A_5 \frac{\partial^2 p}{\partial t^2} &= 0.
\end{aligned} \tag{3.7}$$

Here, we use the notation

$$\begin{aligned}
A_1 &= \frac{FR - MQ}{CR - Q^2} \theta_1 + \frac{\lambda \theta_2}{\lambda + 2\mu}, & A_2 &= \frac{FQ - CM}{CR - Q^2} \theta_1, \\
A_3 &= \frac{\Delta \theta_1}{CR - Q^2} + \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \theta_2, & A_4 &= \frac{Q}{CR - Q^2} \theta_1, \\
A_5 &= \frac{C}{CR - Q^2} \theta_1, & A_6 &= \frac{R\theta_1}{CR - Q^2} + \frac{\theta_2}{\lambda + 2\mu}, & A_7 &= \frac{\theta_1}{L} + \frac{\theta_2}{\mu}, \\
B_1 &= \frac{m_x \bar{\rho} - \rho_f^2}{m_x} \theta_1 + \rho \theta_2, & B_2 &= \frac{\rho_f}{m_x} \theta_1, & B_3 &= \frac{\theta_1}{m_x}, & B_4 &= \bar{\rho} \theta_1 + \rho \theta_2.
\end{aligned} \tag{3.8}$$

Equations (3.6) and (3.7) are equivalent to the equations of Hooke's law

$$\begin{aligned}
\tau_{xx} &= \bar{P} \frac{\partial u_x}{\partial x} + \bar{F} \frac{\partial u_z}{\partial z} + \bar{M} \frac{\partial w_x}{\partial x}, \\
\tau_{zz} &= \bar{F} \frac{\partial u_x}{\partial x} + \bar{C} \frac{\partial u_z}{\partial z} + \bar{Q} \frac{\partial w_x}{\partial x}, \\
-p &= \bar{M} \frac{\partial u_x}{\partial x} + \bar{Q} \frac{\partial u_z}{\partial z} + \bar{R} \frac{\partial w_x}{\partial x}, \\
\tau_{xz} &= \bar{L} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)
\end{aligned} \tag{3.9}$$

and the equations of a continuous medium

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} &= \bar{B}_1 \frac{\partial^2 u_x}{\partial t^2} + \bar{B}_2 \frac{\partial^2 w_x}{\partial t^2}, \\
-\frac{\partial p}{\partial x} &= \bar{B}_2 \frac{\partial^2 u_x}{\partial t^2} + \bar{B}_3 \frac{\partial^2 w_x}{\partial t^2}, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} &= \bar{B}_1 \frac{\partial^2 u_z}{\partial t^2},
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\bar{P} &= \frac{A_3 A_5 A_6 + A_1^2 A_5 + A_2^2 A_6 - 2A_1 A_2 A_4 - A_3 A_4^2}{A_5 A_6 - A_4^2}, \\
\bar{F} &= \frac{A_1 A_5 - A_2 A_4}{A_5 A_6 - A_4^2}, & \bar{M} &= \frac{A_1 A_4 - A_2 A_6}{A_5 A_6 - A_4^2}, \\
\bar{C} &= \frac{A_5}{A_5 A_6 - A_4^2}, & \bar{Q} &= \frac{A_4}{A_5 A_6 - A_4^2}, & \bar{R} &= \frac{A_6}{A_5 A_6 - A_4^2}, & \bar{L} &= \frac{1}{A_7}, \\
\bar{B}_1 = B_4 &= \frac{B_1 B_3 + B_2^2}{B_3} = \bar{\rho} \theta_1 + \rho \theta_2, & \bar{B}_2 &= \frac{B_2}{B_3} = \rho_f, & \bar{B}_3 &= \frac{1}{B_3} = \frac{m_x}{\theta_1}.
\end{aligned} \tag{3.11}$$

The effective model obtained is a special case of the transversely isotropic Biot medium, and the equations describing the former can be represented by relations (1.1) and (1.2) in which (for definiteness) all the coefficients are provided with an additional bar. In order to establish the relation between Eqs. (3.9) and (1.1) and also between relations (3.10) and (1.2), it is necessary to put $\bar{m}_z = \infty$. This equality and relation (1.2) imply that

$$w_z = 0. \tag{3.12}$$

The complete correspondence between the equations in question after passing to the limit in the fourth equation of a continuous medium as $\bar{m}_z \rightarrow 0$ is established with account of equality (3.12). Comparing the coefficients in Eqs. (3.9), (3.10), (1.1), and (1.2), we see that, in the effective Biot medium, the density averaged over the period is equal to $\bar{\rho} \theta_1 + \rho \theta_2$; the density of the fluid phase equals ρ_f ; the tortuosity along

the x axis equals α_x , and the tortuosity along the z axis is equal to ∞ . The porosity f in the effective model is given by the obvious relation $f = f_1\theta_1$.

The equality $\bar{m}_z = \infty$ can be justified by comparing the periodic medium with two Biot layers with the periodic elastic-porous medium. From the third equality in (2.15) and the second relation in (1.3) it follows that

$$m_z = m_{z1}\theta_1 + \frac{\rho f_2 \alpha_{z2} \theta_2}{f_2}. \quad (3.13)$$

In passing from the second porous layer to the elastic layer, one must let f_2 tend to zero. Since $1 \leq \alpha_{z2} < \infty$, we have $m_z \rightarrow \infty$.

4. THE EFFECTIVE MODEL OF A STRATIFIED PERIODIC POROUS-FLUID MEDIUM

Consider the periodic medium (see Fig. 1) every period of which consists of a Biot layer 1 and a fluid layer 2. The porous layers are still described by Eqs. (1.1) and (1.2), in which the displacements and stresses are provided with an additional subscript 1. The displacements u_{x2} , u_{z2} and the pressure p_2 in the fluid layers satisfy the known equations

$$p_2 = -\lambda \left(\frac{\partial u_{x2}}{\partial x} + \frac{\partial u_{z2}}{\partial z} \right), \quad \frac{\partial p_2}{\partial x} = -\rho \frac{\partial^2 u_{x2}}{\partial t^2}, \quad \frac{\partial p_2}{\partial z} = -\rho \frac{\partial^2 u_{z2}}{\partial t^2}, \quad (4.1)$$

where $\lambda = \rho v^2$; ρ is the density of the fluid, and v is the velocity of wave propagation in the latter. On the interfaces between the layers, the conditions

$$u_{z1} + w_{z1} = u_{z2}, \quad \tau_{xx1} = 0, \quad \tau_{zz1} = -p_1 = -p_2 \quad (4.2)$$

are fulfilled. Conditions (4.2) can be regarded as a special case of boundary conditions (1.7). In this case, the porosity of the second layer equals 1, the first condition of (1.7) becomes meaningless, and the second and third conditions in (1.7) are replaced by the first condition in (4.2). Boundary conditions (4.2) take into account the continuity of the total stresses as well as the continuity of the energy stream and the conditions of continuity of the medium.

As in Secs. 2 and 3, in deriving the equations of the effective model, the simplified version of the method of matrix averaging is applied. Since in passing from the original medium to the effective model the interfaces become infinitely close, boundary conditions (4.2) imply that the equalities

$$\tau_{xx1} = 0, \quad \tau_{zz1} = -p_1 \quad (4.3)$$

hold in the whole domain. Simultaneously with taking into account relations (4.3) it is necessary to exclude two equations from relations (1.1), (1.2), describing the Biot medium. One of them is the fourth equation in (1.1) because it involves the derivative $\partial u_x / \partial z$, which becomes meaningless upon passing from the medium to the effective model. Before rejecting the second equation, from the remaining three equations of (1.1) we obtain the following expressions for the derivatives of the displacements $\partial u_{z1} / \partial z$, $\partial w_{z1} / \partial z$ and for the stress τ_{xx1} :

$$\begin{aligned} \frac{\partial u_{z1}}{\partial z} &= \frac{R-Q}{CR-Q^2} \tau_{zz1} - \frac{FR-MQ}{CR-Q^2} \frac{\partial u_{x1}}{\partial x}, \\ \frac{\partial w_{z1}}{\partial z} &= \frac{C-Q}{CR-Q^2} \tau_{zz1} + \frac{FQ-CM}{CR-Q^2} \frac{\partial u_{x1}}{\partial x} - \frac{\partial w_{x1}}{\partial x}, \\ \tau_{xx1} &= \frac{CM+FR-FQ-MQ}{CR-Q^2} \tau_{zz1} + \frac{\Delta}{CR-Q^2} \frac{\partial u_{x1}}{\partial x}. \end{aligned} \quad (4.4)$$

Here, Δ is the determinant of the matrix composed of the coefficients of Hooke's law (1.1), which is determined by (2.2). Since, upon passage to the effective model, the derivatives $\partial u_{z1} / \partial z$ and $\partial w_{z1} / \partial z$ lose their meanings and only the derivative $\partial(u_{z1} + w_{z1}) / \partial z$ remains meaningful, using the first two of relations (4.4), we write the expression

$$\frac{\partial(u_{z1} + w_{z1})}{\partial z} = \frac{R-2Q+C}{CR-Q^2} \tau_{zz1} + \frac{FQ+MQ-CM-FR}{CR-Q^2} \frac{\partial u_{x1}}{\partial x} - \frac{\partial w_{x1}}{\partial x}, \quad (4.5)$$

after which the relations mentioned are discarded. Thus, from the four equations of Hooke's law (1.1) only the third equation from (4.4) and Eq. (4.5) are kept.

In transforming Eqs. (1.2) for subsequent averaging, it is pertinent to note that, with account for relations (4.3), the second and fourth equalities in (1.2) imply the equations

$$\begin{aligned}\frac{\partial \tau_{zz1}}{\partial z} &= \frac{\bar{\rho} m_z - \rho_f^2}{\bar{\rho} - 2\rho_f + m_z} \frac{\partial^2(u_{z1} + w_{z1})}{\partial t^2}, \\ \frac{\partial^2 u_{z1}}{\partial t^2} &= \frac{m_z - \rho_f}{\bar{\rho} - 2\rho_f + m_z} \frac{\partial^2(u_{z1} + w_{z1})}{\partial t^2}, \\ \frac{\partial^2 w_{z1}}{\partial t^2} &= \frac{\bar{\rho} - \rho_f}{\bar{\rho} - 2\rho_f + m_z} \frac{\partial^2(u_{z1} + w_{z1})}{\partial t^2}.\end{aligned}\quad (4.6)$$

Since the displacements u_{z1} and w_{z1} occur only in the second and third equations of (4.6) and are absent in the remaining equations, the former equations can be dropped. Thus, the number of equations for determining the functions $u_{z1} + w_{z1}$, u_{x1} , w_{x1} , τ_{zz1} , and τ_{xx1} is reduced to five. Upon performing necessary transformations, these equations can be written in the form

$$\begin{aligned}\frac{\partial^3(u_{z1} + w_{z1})}{\partial z \partial t^2} &= \frac{R - 2Q + C}{CR - Q^2} \frac{\partial^2 \tau_{zz1}}{\partial t^2} - \frac{1}{m_x} \frac{\partial^2 \tau_{zz1}}{\partial x^2} + \left(\frac{FQ + MQ - CM - FR}{CR - Q^2} + \frac{\rho_f}{m_x} \right) \frac{\partial^3 u_{x1}}{\partial x \partial t^2}, \\ \frac{\partial \tau_{zz1}}{\partial z} &= \frac{\bar{\rho} m_z - \rho_f^2}{\bar{\rho} - 2\rho_f + m_z} \frac{\partial^2(u_{z1} + w_{z1})}{\partial t^2}, \\ \frac{\partial^2 w_{x1}}{\partial t^2} &= \frac{1}{m_x} \frac{\partial \tau_{zz1}}{\partial x} - \frac{\rho_f}{m_x} \frac{\partial^2 u_{x1}}{\partial t^2}, \\ \tau_{xx1} &= \frac{\Delta}{CR - Q^2} \frac{\partial u_{x1}}{\partial x} + \frac{CM + FR - FQ - QR}{CR - Q^2} \tau_{zz1}, \\ (\bar{\rho} m_x - \rho_f^2) \frac{\partial^2 u_{x1}}{\partial t^2} &= m_x \frac{\partial \tau_{xx1}}{\partial x} - \rho_f \frac{\partial \tau_{zz1}}{\partial x}.\end{aligned}\quad (4.7)$$

For the fluid layers, similar equalities (based on (4.1)) are as follows:

$$\begin{aligned}\frac{\partial^3 u_{x2}}{\partial z \partial t^2} &= -\frac{1}{\lambda} \frac{\partial^2 p_2}{\partial t^2} + \frac{1}{\rho} \frac{\partial^2 p_2}{\partial x^2}, \\ \frac{\partial p_2}{\partial z} &= -\rho \frac{\partial^2 u_{x2}}{\partial t^2}, \\ \frac{\partial^2 u_{x2}}{\partial t^2} &= -\frac{1}{\rho} \frac{\partial p_2}{\partial x}.\end{aligned}\quad (4.8)$$

In passing to the fluid layers, two of the functions occurring in Eqs. (4.7), namely, $u_{z1} + w_{z1}$ and τ_{zz1} , are continuously transformed into u_{x2} and $-p_2$, respectively. The functions common to both media will be denoted by u_x and τ_{xx} . In the case considered, the averaging of the remaining functions is carried out in a more sophisticated manner. Comparing the expressions

$$\begin{aligned}w_{x1} &= f_1(u_{x1}^{(1)} - u_{x1}), \\ w_x &= (f_1 \theta_1 + \theta_2) \left[\frac{u_{x1}^{(1)} \theta_1 f_1 + u_{x2} \theta_2}{f_1 \theta_1 + \theta_2} - u_{x1} \right]\end{aligned}$$

for the relative displacements in the layer 1 and in the whole model, we arrive at the relation

$$w_x = \theta_1 w_{x1} + \theta_2 u_{x2} - \theta_2 u_{x1}, \quad (4.9)$$

where $f = f_1\theta_1 + \theta_2$ is the averaged porosity. The total stress τ_{xx} is determined by the equality

$$\tau_{xx} = \tau_{xx1}\theta_1 + \tau_{zz}\theta_2. \quad (4.10)$$

In passing to the effective model, the displacement u_{x1} preserves its physical meaning, and the expression for it remains unchanged. Therefore,

$$u_x = u_{x1}. \quad (4.11)$$

Applying the approach used in Secs. 2 and 3, we write the equations

$$\begin{aligned} \frac{\partial^3 u_z}{\partial z \partial t^2} &= A_1 \frac{\partial^2 \tau_{zz}}{\partial t^2} - \left(\frac{\theta_1}{m_x} + \frac{\theta_2}{\rho} \right) \frac{\partial^2 \tau_{zz}}{\partial x^2} + \theta_1 \left(A_2 + \frac{\rho_f}{m_x} \right) \frac{\partial^3 u_x}{\partial x \partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial z} &= B \frac{\partial^2 u_z}{\partial t^2}, \\ \frac{\partial^2 w_x}{\partial t^2} &= \left(\frac{\theta_1}{m_x} + \frac{\theta_2}{\rho} \right) \frac{\partial \tau_{zz}}{\partial x} - \left(\frac{\theta_1 \rho_f}{m_x} + \theta_2 \right) \frac{\partial^2 u_x}{\partial t^2}, \\ \tau_{xx} &= \theta_1 A_3 \frac{\partial u_x}{\partial x} + (\theta_2 - \theta_1 A_2) \tau_{zz}, \\ \theta_1 (m_x \bar{\rho} - \rho_f^2) \frac{\partial^2 u_x}{\partial t^2} &= m_x \frac{\partial \tau_{xx}}{\partial x} - (m_x \theta_2 + \rho_f \theta_1) \frac{\partial \tau_{zz}}{\partial x}, \end{aligned} \quad (4.12)$$

where the following notation is used:

$$\begin{aligned} A_1 &= \frac{R - 2Q + C}{CR - Q^2} \theta_1 + \frac{\theta_2}{\lambda}, \quad A_2 = \frac{FQ + MQ - CM - FR}{CR - Q^2}, \\ A_3 &= \frac{\Delta}{CR - Q^2}, \quad B = \frac{\bar{\rho} m_x - \rho_f^2}{\bar{\rho} - 2\rho_f + m_x} \theta_1 + \rho \theta_2. \end{aligned} \quad (4.13)$$

Equations (4.12) are equivalent to Hooke's equations

$$\begin{aligned} \tau_{xx} &= H_1 \frac{\partial u_x}{\partial x} + H_2 \left(\frac{\partial u_z}{\partial z} + \frac{\partial w_x}{\partial x} \right), \\ \tau_{zz} &= H_2 \frac{\partial u_x}{\partial x} + H_3 \left(\frac{\partial u_z}{\partial z} + \frac{\partial w_x}{\partial x} \right) \end{aligned} \quad (4.14)$$

and the equations of a continuous medium

$$\begin{aligned} \frac{\partial \tau_{xx}}{\partial x} &= \bar{\rho} \frac{\partial^2 u_x}{\partial t^2} + \bar{\rho}_f \frac{\partial^2 w_x}{\partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial x} &= \bar{\rho}_f \frac{\partial^2 u_x}{\partial t^2} + \bar{m}_x \frac{\partial^2 w_x}{\partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial z} &= B \frac{\partial^2 u_z}{\partial t^2}. \end{aligned} \quad (4.15)$$

In (4.14) and (4.15), we use the notation

$$\begin{aligned} H_1 &= A_3 \theta_1 + \frac{(\theta_2 - A_2 \theta_1)^2}{A_1}, \quad H_2 = \frac{\theta_2 - \theta_1 A_2}{A_1}, \quad H_3 = \frac{1}{A_1}, \\ \bar{\rho} &= \frac{\rho \bar{\rho} \theta_1^2 + (m_x \bar{\rho} - \rho_f^2 + 2\rho \rho_f) \theta_1 \theta_2 + \rho m_x \theta_2^2}{\rho \theta_1 + m_x \theta_2}, \\ \bar{\rho}_f &= \rho \frac{\rho_f \theta_1 + m_x \theta_2}{\rho \theta_1 + m_x \theta_2}, \quad \bar{m}_x = \frac{\rho m_x}{\rho \theta_1 + m_x \theta_2}. \end{aligned} \quad (4.16)$$

The effective model of the medium under consideration is a generalized transversely isotropic Biot medium. In order to prove this claim, we consider Eqs. (1.1) and (2.14) in which, for clarity, the coefficients are provided with an additional bar. We will find the values of the parameters for which Eqs. (1.1) and (2.14) turn into Eqs. (4.14) and (4.15). The assumptions $\bar{L} = 0$, $\bar{F} = \bar{M}$, $\bar{C} = \bar{Q} = \bar{R}$, and $\bar{m}_z = \infty$ lead to the equalities

$$\tau_{xx} = 0, \quad \tau_{zz} = -p, \quad w_z = 0, \quad (4.17)$$

which hold on the entire interval $0 \leq z \leq H$. Under these assumptions, the medium under consideration is described by the equations of Hooke's law

$$\begin{aligned} \tau_{xx} &= \bar{P} \frac{\partial u_x}{\partial x} + \bar{F} \left(\frac{\partial u_z}{\partial z} + \frac{\partial w_x}{\partial x} \right), \\ \tau_{zz} &= \bar{F} \frac{\partial u_x}{\partial x} + \bar{C} \left(\frac{\partial u_z}{\partial z} + \frac{\partial w_x}{\partial x} \right) \end{aligned} \quad (4.18)$$

and the equations of a continuous medium

$$\begin{aligned} \frac{\partial \tau_{xx}}{\partial x} &= \bar{\rho}_x \frac{\partial^2 u_x}{\partial t^2} + \bar{\rho}_f \frac{\partial^2 w_x}{\partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial x} &= \bar{\rho}_f \frac{\partial^2 u_x}{\partial t^2} + \bar{m}_x \frac{\partial^2 w_x}{\partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial z} &= \bar{\rho}_z \frac{\partial^2 u_z}{\partial t^2}. \end{aligned} \quad (4.19)$$

The coefficients in Eqs. (4.18) and (4.19) are determined by comparing these equations with relations (4.14) and (4.15). As a result, for the parameters of the generalized transversely isotropic Biot medium we obtain the values

$$\begin{aligned} \bar{L} &= 0, \quad \bar{P} = H_1, \quad \bar{F} = \bar{M} = H_2, \quad \bar{C} = \bar{Q} = \bar{R} = H_3, \quad \bar{\rho}_x = \bar{\rho}, \\ \bar{\rho}_{fx} &= \rho_f, \quad \bar{m}_x = m_x, \quad \bar{\rho}_z = B, \quad \bar{m}_z = \infty. \end{aligned} \quad (4.20)$$

Note that the parameter ρ_{fx} proves to be an arbitrary finite quantity.

In conclusion, we point out that the effective model obtained is a generalization of the effective model of a stratified elastic-fluid medium. In order to establish the interrelation between the elastic-fluid models, we use the equalities

$$w_x = f(u_x^{(1)} - u_x), \quad \tau_{xx} = \sigma_{xx}^{(2)} + \tau_{zx}. \quad (4.21)$$

The first of them is just the second equality in (1.5), whereas the second one corresponds to the decomposition of the total stress τ_{xx} into the stresses in the phases. Here, $\sigma_{xx}^{(2)}$ is the averaged stress in the elastic phase. Taking into account formulas (4.21), we can replace Eqs. (4.18) and (4.19) by the equalities

$$\begin{aligned} \sigma_{xx}^{(2)} &= (\bar{P} - 2\bar{F}f + \bar{C}f^2) \frac{\partial u_x}{\partial x} + (\bar{F} - f\bar{C}) \left(\frac{\partial u_z}{\partial z} + f \frac{\partial u_x^{(1)}}{\partial x} \right), \\ \tau_{zz} &= (\bar{F} - f\bar{C}) \frac{\partial u_x}{\partial x} + \bar{C} \left(\frac{\partial u_z}{\partial z} + f \frac{\partial u_x^{(1)}}{\partial x} \right), \\ \frac{\partial \sigma_{xx}^{(2)}}{\partial x} &= (\bar{\rho}_x - 2\bar{\rho}_f f + \bar{m}_x f^2) \frac{\partial^2 u_x}{\partial t^2} + f(\bar{\rho}_f - f\bar{m}_x) \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial x} &= (\bar{\rho}_f - f\bar{m}_x) \frac{\partial^2 u_x}{\partial t^2} + f\bar{m}_x \frac{\partial^2 u_x^{(1)}}{\partial t^2}, \\ \frac{\partial \tau_{zz}}{\partial z} &= \bar{\rho}_z \frac{\partial^2 u_z}{\partial t^2}. \end{aligned} \quad (4.22)$$

Under the assumption that either $\bar{\rho}_f = \bar{m}_x f$ or $\bar{\alpha}_x = 1$, Eqs. (4.22) coincide with the equations of the effective model of a layered elastic-fluid medium [5, 13]. The latter model, in turn, is a special case of the ordinary transversely isotropic Biot model [14].

This work was supported by the Russian Foundation for Basic Research under grant Nos. 96-01-00666 and 96-05-66207.

Translated by L. A. Molotkov.

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