

BODY FORCES EQUIVALENT TO JUMPS OF STRESSES AND DISPLACEMENTS IN THE BIOT MODEL OF AN ANISOTROPIC POROUS MEDIUM

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Formulas for the body forces equivalent to given jumps of stresses and displacements in the anisotropic Biot model are obtained. Similarly to the case of elastic media, Betti's identity, the reciprocity relations, and the representation integrals for a porous media are derived first. The equivalent body forces are obtained by applying the representation integrals to a volume with surfaces of discontinuity. Some physical consequences are indicated, and examples of equivalent body forces for the isotropic Biot model are provided. Bibliography: 7 titles.

INTRODUCTION

Internal seismic sources in elastic media can be described [1, 2] either as body forces in the wave equations or as jumps of stresses and displacements on possibly fictitious interfaces. These two approaches are interrelated in the following way [2]. For any distribution of the jumps of displacements and stresses, there always exists a combination of body forces that results in the same wave field. Formulas for these equivalent body forces are provided in [2].

This relation between the two descriptions of sources is of importance because, first, it permits one to obtain the so-called Green's (Stokes') functions in different ways and, second, it provides the possibility of analyzing intricate wave fields excited by sources described in terms of jumps of stresses and displacements by decomposing these sources into a combination of elementary ones. This possibility is especially important in analyzing wave fields generated by earthquakes, which are modeled by slips (jumps of displacements) along faults (interfaces) [2].

Sources in the more sophisticated Biot model of porous rocks are described analogously [3, 4]. This model is frequently used in exploration seismology when considering wave propagation in fractured and porous media. For this reason, the explicit relationship between the two descriptions of sources in Biot's model is of considerable interest.

In this paper, based on the technique developed in [2] for the case of elastic media, the expressions for the body forces equivalent to given jumps of stresses and displacements are derived for the Biot model of an anisotropic fluid-saturated medium.

1. BETTI'S IDENTITY AND THE RECIPROCITY RELATIONS

The equations of Hooke's law for the anisotropic Biot model can be written in the form [5]

$$\sigma_{ij} = A_{ij}^{\mu\nu} e_{\mu\nu} + Q_{ij}\theta, \quad \sigma = Q_{ij}e_{ij} + R\theta. \quad (1.1)$$

Here, σ and σ_{ij} are the stresses in the fluid (first) and solid (second) phases, respectively, averaged over the total bulk volume; e_{ij} are the skeleton deformations, determined by the formula

$$e_{ij} = \begin{cases} \frac{\partial u_i^{(2)}}{\partial x_i}, & i = j, \\ \frac{\partial u_i^{(2)}}{\partial x_j} + \frac{\partial u_j^{(2)}}{\partial x_i}, & i \neq j; \end{cases} \quad (1.2)$$

θ is the divergence of displacements in the fluid phase. As pointed out in [6], $\bar{u}^{(2)}$ is the vector of skeleton displacements averaged over the volume of the solid phase, and $\bar{u}^{(1)}$ is the displacement vector of the fluid averaged over the volume of the fluid phase. This conclusion becomes obvious if one considers the effective model of a layered fluid-solid medium, which is a special case of Biot's model [6]. In Eq. (1.1) and henceforward, summation from 1 to 3 over repeated indices is implied. The elastic coefficients $A_{ij}^{\mu\nu}$, Q_{ij} , and R form a positive-definite 7×7 -matrix and satisfy the relations of symmetry

$$A_{ij}^{\mu\nu} = A_{ji}^{\mu\nu} = A_{ij}^{\nu\mu} = A_{ji}^{\nu\mu}, \quad Q_{ij} = Q_{ji}. \quad (1.3)$$

As is easily seen, these symmetry relations imply Betti's identity for Biot's medium

$$\sum_{i \leq j} \sigma_{ij} e_{ij}^0 + \sigma \theta^0 = \sum_{i \leq j} \sigma_{ij}^0 e_{ij} + \sigma^0 \theta, \quad (1.4)$$

where σ_{ij} , σ , e_{ij} , θ and σ_{ij}^0 , σ^0 , e_{ij}^0 , θ^0 are two sets of stresses and deformations resulting from the application of different external forces to the same porous medium.

In order to obtain the relations of the reciprocity theorem, we use the identity [7]

$$\operatorname{div} \left(u_k^{(2)} \bar{\sigma}_k \right) + \operatorname{div} \left(\sigma \bar{u}^{(1)} \right) = \bar{u}^{(2)} \operatorname{Div} \sigma_{ik} + \sum_{i \leq k} \sigma_{ik} e_{ik} + \sigma \theta + \bar{u}^{(1)} \operatorname{grad} \sigma, \quad (1.5)$$

which is valid for arbitrary vectors $\bar{u}^{(1)}$, $\bar{u}^{(2)}$, where e_{ik} are the corresponding deformations (see (1.2)); $\bar{\sigma}_k = \sigma_{kj} \bar{i}_j$; σ_{kj} is a symmetric table, and $\bar{i}_j$ is the j th unit coordinate vector. In Eq. (1.5), we use the notation [7]

$$\operatorname{Div} \sigma_{ik} = \bar{i}_j \operatorname{div} \bar{\sigma}_j. \quad (1.6)$$

Let $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ be the displacement fields in a volume V corresponding to certain body forces, boundary conditions on the surface S surrounding V , and initial conditions at $t = 0$. Further, let $\bar{v}^{(1)}$ and $\bar{v}^{(2)}$ be the displacement fields in the same volume V that correspond to some other body forces and other boundary and initial conditions. By σ_{ik}^0 , σ^0 and σ_{ik} , σ we denote the stresses from Eq. (1.1) corresponding to the fields $\bar{u}^{(1)}$, $\bar{u}^{(2)}$ and $\bar{v}^{(1)}$, $\bar{v}^{(2)}$, respectively. We write identity (1.5) for the two sets σ_{ik} , σ , $\bar{v}^{(1)}$, $\bar{v}^{(2)}$ and σ_{ik}^0 , σ^0 , $\bar{u}^{(1)}$, $\bar{u}^{(2)}$ and subtract the second relation obtained in this way from the first one. Applying Betti's formula (1.4), we derive

$$\operatorname{div} \left(v_k^{(2)} \bar{\sigma}_k - u_k^{(2)} \bar{\sigma}_k^0 \right) + \operatorname{div} \left(\sigma \bar{v}^{(1)} - \sigma^0 \bar{u}^{(1)} \right) = \bar{v}^{(2)} \operatorname{Div} \sigma_{ik} - \bar{u}^{(2)} \operatorname{Div} \sigma_{ik}^0 + \bar{v}^{(1)} \operatorname{grad} \sigma - \bar{u}^{(1)} \operatorname{grad} \sigma^0. \quad (1.7)$$

In order to transform the right-hand side of (1.7), we use the equilibrium equations for Biot's model, which, in the above notation, are of the form

$$\begin{aligned} \operatorname{Div} \sigma_{ik} + \bar{f}^{(2)} &= \rho_{11} \frac{\partial^2 \bar{u}^{(2)}}{\partial t^2} + \rho_{12} \frac{\partial^2 \bar{u}^{(1)}}{\partial t^2}, \\ \operatorname{grad} \sigma + \bar{f}^{(1)} &= \rho_{12} \frac{\partial^2 \bar{u}^{(2)}}{\partial t^2} + \rho_{22} \frac{\partial^2 \bar{u}^{(1)}}{\partial t^2}. \end{aligned} \quad (1.8)$$

Here, ρ_{11} , ρ_{22} , ρ_{12} are 3×3 -matrices. Assume that the body forces $\bar{f}^{(1)}$, $\bar{f}^{(2)}$ and $\bar{g}^{(1)}$, $\bar{g}^{(2)}$ correspond to the fields $\bar{u}^{(1)}$, $\bar{u}^{(2)}$ and $\bar{v}^{(1)}$, $\bar{v}^{(2)}$, respectively. Substituting the expressions for $\operatorname{Div} \sigma_{ik}$ and $\operatorname{grad} \sigma$ implied by (1.8) into Eqs. (1.7), we arrive at the first differential relation of reciprocity:

$$\begin{aligned} & \frac{\partial}{\partial x_k} \left(\bar{\sigma}_k^0 \bar{u}^{(2)} - \bar{\sigma}_k \bar{v}^{(2)} + \sigma^0 u_k^{(1)} - \sigma v_k^{(1)} \right) + \rho_{11} \frac{\partial}{\partial t} \left(\bar{v}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial t} - \bar{u}^{(2)} \frac{\partial \bar{v}^{(2)}}{\partial t} \right) \\ & + \rho_{12} \frac{\partial}{\partial t} \left(\bar{v}^{(2)} \frac{\partial \bar{u}^{(1)}}{\partial t} - \bar{u}^{(1)} \frac{\partial \bar{v}^{(2)}}{\partial t} \right) + \rho_{12} \frac{\partial}{\partial t} \left(\bar{u}^{(2)} \frac{\partial \bar{v}^{(1)}}{\partial t} - \bar{v}^{(1)} \frac{\partial \bar{u}^{(2)}}{\partial t} \right) \\ & + \rho_{22} \frac{\partial}{\partial t} \left(\bar{v}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial t} - \bar{u}^{(1)} \frac{\partial \bar{v}^{(1)}}{\partial t} \right) = \bar{f}^{(2)} \bar{v}^{(2)} - \bar{g}^{(2)} \bar{u}^{(2)} + \bar{f}^{(1)} \bar{v}^{(1)} - \bar{g}^{(1)} \bar{u}^{(1)}. \end{aligned} \quad (1.9)$$

In transforming the left-hand side of (1.9), we used the relation $\bar{\sigma}_k = \sigma_{kj} \bar{i}_j$ and the symmetry of σ_{kj} . Integrating over the volume V and taking into account the equality

$$\bar{\sigma}_n \bar{u} = (\bar{\sigma}_1 u_1 + \bar{\sigma}_2 u_2 + \bar{\sigma}_3 u_3) \bar{n}, \quad (1.10)$$

where $\bar{\sigma}_n$ is the stress vector acting at a point of the surface with normal vector $\bar{n}$, we rewrite (1.9) in integral form

$$\begin{aligned} \iint_S \left(\bar{\sigma}_n^0 \bar{u}^{(2)} - \bar{\sigma}_n \bar{v}^{(2)} + \sigma^0 \bar{n} \bar{u}^{(1)} - \sigma \bar{n} \bar{v}^{(1)} \right) dS = \iiint_V \left\{ \bar{f}^{(2)} \bar{v}^{(2)} - \bar{g}^{(2)} \bar{u}^{(2)} + \bar{f}^{(1)} \bar{v}^{(1)} - \bar{g}^{(1)} \bar{u}^{(1)} \right. \\ \left. - \rho_{11} \frac{\partial}{\partial t} \left(\bar{v}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial t} - \bar{u}^{(2)} \frac{\partial \bar{v}^{(2)}}{\partial t} \right) - \rho_{12} \frac{\partial}{\partial t} \left(\bar{v}^{(2)} \frac{\partial \bar{u}^{(1)}}{\partial t} - \bar{u}^{(1)} \frac{\partial \bar{v}^{(2)}}{\partial t} \right) \right. \\ \left. - \rho_{12} \frac{\partial}{\partial t} \left(\bar{u}^{(2)} \frac{\partial \bar{v}^{(1)}}{\partial t} - \bar{v}^{(1)} \frac{\partial \bar{u}^{(2)}}{\partial t} \right) - \rho_{22} \frac{\partial}{\partial t} \left(\bar{v}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial t} - \bar{u}^{(1)} \frac{\partial \bar{v}^{(1)}}{\partial t} \right) \right\} dV. \end{aligned} \quad (1.11)$$

Formula (1.11) does not involve the initial conditions for the fields $\bar{u}^{(1)}$, $\bar{u}^{(2)}$ and $\bar{v}^{(1)}$, $\bar{v}^{(2)}$. Therefore, it is also applicable to the case where $\bar{\sigma}_n$, σ , $\bar{u}^{(1)}$, $\bar{u}^{(2)}$, $\bar{f}^{(1)}$, and $\bar{f}^{(2)}$ are estimated at a time t_1 , whereas $\bar{\sigma}_n^0$, σ^0 , $\bar{v}^{(1)}$, $\bar{v}^{(2)}$, $\bar{g}^{(1)}$, and $\bar{g}^{(2)}$ are estimated at a time t_2 . Furthermore, let $t_1 = t$ and let $t_2 = \tau - t$. If we assume that, in the volume V , all of the displacement fields are zero for $t \leq t_0$, then the convolutions of all the terms of the form

$$\int_{-\infty}^{+\infty} \rho_{lm} \frac{\partial}{\partial t} \left(\bar{v}^{(j)}(\tau - t) \frac{\partial \bar{u}^{(i)}}{\partial t}(t) - \bar{u}^{(i)}(t) \frac{\partial \bar{v}^{(j)}}{\partial t}(\tau - t) \right) dt$$

vanish. Upon integrating with respect to t , formula (1.11) for the displacement fields vanishing for $t \leq t_0$ takes the form

$$\begin{aligned} \int_{-\infty}^{+\infty} dt \int \int_V \left\{ \bar{f}^{(2)}(t) \bar{v}^{(2)}(\tau - t) - \bar{g}^{(2)}(\tau - t) \bar{u}^{(2)}(t) + \bar{f}^{(1)}(t) \bar{v}^{(1)}(\tau - t) - \bar{g}^{(1)}(\tau - t) \bar{u}^{(1)}(t) \right\} dV \\ = \int_{-\infty}^{+\infty} dt \iint_S \left\{ \bar{\sigma}_n^0(\tau - t) \bar{u}^{(2)}(t) - \bar{\sigma}_n(t) \bar{v}^{(2)}(\tau - t) + \sigma^0(\tau - t) \bar{n} \bar{u}^{(1)}(t) - \sigma(t) \bar{n} \bar{v}^{(1)}(\tau - t) \right\} dS. \end{aligned} \quad (1.12)$$

This is a very important integral reciprocity relation, which permits one to obtain various formulations of the space-time reciprocity.

2. REPRESENTATION INTEGRALS AND EQUIVALENT BODY FORCES

In order to obtain representation integrals, which will enable us to determine the equivalent body forces, it is necessary to take Green's function as one of the displacement fields in (1.12). For the anisotropic Biot medium, it is convenient to represent Green's function as a sum of two terms, $G_{il}^{(1,2)}(\bar{x}, t; \bar{\xi}, t_0)$ and $W_i^{(1,2)}(\bar{x}, t; \bar{\xi}, t_0)$. In the volume V , each of the terms ($u_i^{(1)} = G_{il}^{(1)}$, $u_i^{(2)} = G_{il}^{(2)}$ and $u_i^{(1)} = W_i^{(1)}$, $u_i^{(2)} = W_i^{(2)}$) satisfies Eqs. (1.1) and (1.8), where $\bar{f}^{(1)}$ and $\bar{f}^{(2)}$ are different external forces with time dependence $\delta(t - t_0)$, applied to the point $\bar{\xi}$. These functions, as well as their first derivatives, are assumed to vanish for $t \leq t_0$ and $\bar{x} \neq \bar{\xi}$. On the surface S with normal vector $\bar{n}$, the homogeneous boundary conditions are imposed. For a porous medium, this means that two relations, one from each of the pairs

$$\bar{u}^{(2)} = 0, \quad \bar{\sigma}_n = 0; \quad \bar{u}^{(1)} \bar{n} = 0, \quad \sigma = 0, \quad (2.1)$$

are satisfied. The first term, $G_{il}^{(1,2)}(\bar{x}, t; \bar{\xi}, t_0)$, is related to the displacement field corresponding to the unit force $\bar{f}^{(1)} = 0$, $\bar{f}_i^{(2)} = \delta_{il} \delta(\bar{x} - \bar{\xi}) \delta(t - t_0)$, applied to the skeleton and acting along the l th coordinate

direction, where δ_{il} is the Kronecker symbol. The second term, $W_i^{(1,2)}(\vec{x}, t; \vec{\xi}, t_0)$, corresponds to the field excited by the unit center of pressure $\vec{f}^{(1)} = \delta(t - t_0) \text{grad}(\vec{x} - \vec{\xi})$, $\vec{f}^{(2)} = 0$ in the fluid. Such a definition of the second term of the Green's function is related to the fact that, in a fluid, it is not allowed to choose a point source of the type of a directed force [3]. Sources in fluids ($\vec{f}^{(1)}$) must necessarily be irrotational, and $\text{grad} \delta(\vec{x})$ is taken as an elementary irrotational action.

Taking $G_{il}^{(1,2)}(\vec{x}, t; \vec{\xi}, t_0)$ and $W_i^{(1,2)}(\vec{x}, t; \vec{\xi}, t_0)$ as the fields $\vec{v}_i^{(1)}$, $\vec{v}_i^{(2)}$ and setting $t_0 = 0$, we arrive at the following two representation integrals:

$$\begin{aligned} u_i^{(2)}(\vec{\xi}, \tau) = & \int_{-\infty}^{+\infty} dt \iiint_V \left\{ f_i^{(2)}(\vec{x}, t) G_{il}^{(2)}(\vec{x}, \tau - t; \vec{\xi}, 0) + f_i^{(1)}(\vec{x}, t) G_{il}^{(1)}(\vec{x}, \tau - t; \vec{\xi}, 0) \right\} dV \\ & + \int_{-\infty}^{+\infty} dt \iint_S \left\{ G_{il}^{(2)}(\vec{x}, \tau - t; \vec{\xi}, 0) (\vec{\sigma}_n(\vec{x}, t))_i + G_{il}^{(1)}(\vec{x}, \tau - t; \vec{\xi}, 0) \sigma(\vec{x}, t) n_i \right. \\ & \left. - \vec{u}^{(2)}(\vec{x}, t) \vec{\sigma}_n^0(\vec{x}, \tau - t) - \vec{u}^{(1)}(\vec{x}, t) \sigma^0(\vec{x}, \tau - t) \vec{n} \right\} dS, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \theta(\vec{\xi}, \tau) = & \int_{-\infty}^{+\infty} dt \iiint_V \left\{ f_i^{(2)}(\vec{x}, t) W_i^{(2)}(\vec{x}, \tau - t; \vec{\xi}, 0) \right. \\ & \left. + f_i^{(1)}(\vec{x}, t) W_i^{(1)}(\vec{x}, \tau - t; \vec{\xi}, 0) \right\} dV + \int_{-\infty}^{+\infty} dt \iint_S \left\{ W_i^{(2)}(\vec{x}, \tau - t; \vec{\xi}, 0) (\vec{\sigma}_n(\vec{x}, t))_i \right. \\ & \left. + W_i^{(1)}(\vec{x}, \tau - t; \vec{\xi}, 0) \sigma(\vec{x}, t) n_i - \vec{u}^{(2)}(\vec{x}, t) \vec{\sigma}_n^0(\vec{x}, \tau - t) - \vec{u}^{(1)}(\vec{x}, t) \sigma^0(\vec{x}, \tau - t) \vec{n} \right\} dS. \end{aligned} \quad (2.3)$$

Here, $(\vec{\sigma}_n)_k = \vec{\sigma}_k \vec{n}$ is the k th component of the vector $\vec{\sigma}_n = (\vec{\sigma}_k \vec{n}) \vec{i}_k$ of normal stresses, where $\vec{\sigma}_n^0$ and σ^0 are the stresses corresponding to the displacement fields $G_{il}^{(1)}$, $G_{il}^{(2)}$ in (2.2) and $W_i^{(1)}$, $W_i^{(2)}$ in (2.3), as above.

Relations (2.2) and (2.3) express the displacements of the skeleton $\vec{u}^{(2)}$ and the divergence θ of the fluid in terms of the body forces $\vec{f}^{(1)}$ and $\vec{f}^{(2)}$, acting inside V , of the stresses $\vec{\sigma}_n$ and σ , and of the displacements $\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ on the surface S .

We apply formulas (2.2) and (2.3) to a volume V the surface of which consists of an external boundary surface S and of the two sides Σ^+ and Σ^- of one and the same internal interface Σ , on which stresses and displacements have jumps. Since the equations of motion are not satisfied immediately on Σ , these formulas can only be applied to the interior part of the surface $S + \Sigma^+ + \Sigma^-$. Assume also that, on S , the homogeneous boundary conditions (2.1) are fulfilled. Then the relations of the representation theorem can be rewritten in the following form:

$$\begin{aligned} u_i^{(2)}(\vec{x}, t) = & \int_{-\infty}^{+\infty} d\tau \iiint_V \left\{ f_i^{(2)}(\vec{\eta}, \tau) G_{il}^{(2)}(\vec{\eta}, t - \tau; \vec{x}, 0) + f_i^{(1)}(\vec{\eta}, \tau) G_{il}^{(1)}(\vec{\eta}, t - \tau; \vec{x}, 0) \right\} dV(\vec{\eta}) \\ & + \int_{-\infty}^{+\infty} d\tau \iint_{\Sigma} \left\{ \left[\vec{u}^{(2)}(\vec{\xi}, \tau) \right] \vec{\sigma}_n^0(\vec{\xi}, t - \tau) + \left[\vec{u}^{(1)}(\vec{\xi}, \tau) \right] \sigma^0(\vec{\xi}, t - \tau) \vec{n} \right. \\ & \left. - \left[(\vec{\sigma}_n(\vec{\xi}, \tau))_i \right] G_{il}^{(2)}(\vec{\xi}, t - \tau; \vec{x}, 0) - \left[\vec{\sigma}(\vec{\xi}, \tau) \right] n_i G_{il}^{(1)}(\vec{\xi}, t - \tau; \vec{x}, 0) \right\} d\Sigma(\vec{\xi}), \end{aligned} \quad (2.4)$$

$$\begin{aligned} \theta(\vec{x}, t) = & \int_{-\infty}^{+\infty} d\tau \iiint_V \left\{ f_i^{(2)}(\vec{\eta}, \tau) W_i^{(2)}(\vec{\eta}, t - \tau; \vec{x}, 0) + f_i^{(1)}(\vec{\eta}, \tau) W_i^{(1)}(\vec{\eta}, t - \tau; \vec{x}, 0) \right\} dV(\vec{\eta}) \\ & + \int_{-\infty}^{+\infty} d\tau \iint_{\Sigma} \left\{ \left[\vec{u}^{(2)}(\vec{\xi}, \tau) \right] \vec{\sigma}_n^0(\vec{\xi}, t - \tau) + \left[\vec{u}^{(1)}(\vec{\xi}, \tau) \right] \sigma^0(\vec{\xi}, t - \tau) \vec{n} \right. \\ & \left. - \left[(\vec{\sigma}_n(\vec{\xi}, \tau))_i \right] W_i^{(2)}(\vec{\xi}, t - \tau; \vec{x}, 0) - \left[\sigma(\vec{\xi}, \tau) \right] n_i W_i^{(1)}(\vec{\xi}, t - \tau; \vec{x}, 0) \right\} d\Sigma(\vec{\xi}). \end{aligned} \quad (2.5)$$

Note that, as compared to (2.2), (2.3), in Eqs. (2.4), (2.5) we have interchanged the symbols $\vec{x}$ and t with the symbols $\vec{\xi}$ and τ , so that $(\vec{x}, t)$ is an arbitrary space-time point. In integrating over the volume V and the surface Σ , arbitrary points of them are denoted by $\vec{\eta}$ and $\vec{\xi}$, respectively. Square brackets denote the difference between the values on Σ^+ and Σ^- .

Analyzing Eqs. (2.4) and (2.5), we can immediately determine the body forces that result in the same wave fields as the jumps of displacements and stresses in Eqs. (2.4), (2.5). These forces are called the equivalent body forces.

(1) To jumps of stresses in the skeleton the body force

$$\vec{f}^{(1)} = 0, \quad \vec{f}^{(2)}(\vec{\eta}, \tau) = - \iint_{\Sigma} [\vec{\sigma}_n(\vec{\xi}, \tau)] \delta(\vec{\eta} - \vec{\xi}) d\Sigma(\vec{\xi}), \quad (2.6)$$

applied only to the skeleton, is equivalent.

(2) To jumps of pressure in the fluid there corresponds the force

$$\vec{f}^{(1)}(\vec{\eta}, \tau) = - \iint_{\Sigma} [\sigma(\vec{\xi}, \tau)] \vec{n} \delta(\vec{\eta} - \vec{\xi}) d\Sigma(\vec{\xi}), \quad \vec{f}^{(2)} = 0, \quad (2.7)$$

applied to the fluid phase.

(3) Jumps of displacements in the solid phase are equivalent to the forces

$$\begin{aligned} \vec{f}^{(1)}(\vec{\eta}, \tau) &= \text{grad } \Phi(\vec{\eta}, \tau), \quad \Phi(\vec{\eta}, \tau) = - \iint_{\Sigma} [u_k^{(2)}(\vec{\xi}, \tau)] Q_{ik} n_i \delta(\vec{\eta} - \vec{\xi}) d\Sigma(\vec{\xi}), \\ f_j^{(2)}(\vec{\eta}, \tau) &= - \iint_{\Sigma} [u_k^{(2)}(\vec{\xi}, \tau)] A_{ik}^{jl} n_i \frac{\partial}{\partial \eta_l} \delta(\vec{\eta} - \vec{\xi}) d\Sigma(\vec{\xi}). \end{aligned} \quad (2.8)$$

In deriving (2.8), we have used Hooke's law (1.1) in addition to (2.4) and (2.5).

(4) To jumps of displacements in the fluid phase the forces

$$\begin{aligned} \vec{f}^{(1)}(\vec{\eta}, \tau) &= \text{grad } \Phi(\vec{\eta}, \tau), \quad \Phi(\vec{\eta}, \tau) = - \iint_{\Sigma} R [\vec{u}^{(1)}(\vec{\xi}, \tau)] \vec{n} \delta(\vec{\eta} - \vec{\xi}) d\Sigma(\vec{\xi}), \\ f_j^{(2)}(\vec{\eta}, \tau) &= - \iint_{\Sigma} [u_k^{(1)}(\vec{\xi}, \tau)] n_k Q_{jl} \frac{\partial}{\partial \eta_l} \delta(\vec{\eta} - \vec{\xi}) d\Sigma(\vec{\xi}) \end{aligned} \quad (2.9)$$

are equivalent.

3. PHYSICAL CONSEQUENCES AND EXAMPLES

Relations (2.7)–(2.9) imply the following physical consequences.

I. A point jump of the normal stress in the solid across the plane $\xi_3 = 0$

$$[\sigma_{33}] = F(t) \delta(\xi_1) \delta(\xi_2), \quad [\sigma_{13}] = [\sigma_{23}] = 0 \quad (3.1)$$

results in the same displacement field as the unit force

$$\vec{f}^{(1)} = 0, \quad \vec{f}^{(2)}(\vec{\eta}, \tau) = -(0, 0, F(\tau))^T \delta(\eta_1) \delta(\eta_2) \delta(\eta_3), \quad (3.2)$$

applied to the skeleton. This is analogous to the case of an elastic medium.

II. A point jump of pressure in the fluid phase across the plane $\xi_3 = 0$

$$[\sigma] = F(t) \delta(\xi_1) \delta(\xi_2) \quad (3.3)$$

is not allowed. We prove this claim by contradiction. Suppose that (3.3) is given. Then, in accordance with (2.7), this jump is equivalent to the body force

$$\vec{f}^{(1)}(\vec{\eta}, \tau) = -(0, 0, F(\tau))^T \delta(\eta_1) \delta(\eta_2) \delta(\eta_3), \quad (3.4)$$

applied to the fluid. The force $\vec{f}^{(1)}$ has a nontrivial rotational part ($\text{rot } \vec{f}^{(1)} \neq 0$), and, according to [3], such a source is not allowed in the fluid phase.

Nevertheless, the above assertion does not mean that any jump of pressure in the fluid phase is prohibited. Examples of physically meaningful sources of this type are provided by sources determined by constant jumps of pressure on an infinitely small sphere [4] or on an infinite plane, which correspond to a center of pressure and to a piston action, respectively. For such jumps, the equivalent forces $\vec{f}^{(1)}$ are irrotational.

III. For jumps of displacements, the equivalent body forces $\vec{f}^{(1)}$ in the fluid phase are always irrotational, as follows from formulas (2.8) and (2.9). Therefore, for Biot's model, arbitrary discontinuities of displacements (in contrast to jumps of pressure in the fluid phase) are always physically correct.

IV. As is implied by (2.9), which involves the scalar product $[\vec{u}^{(1)}] \vec{n}$ as a multiplier, only the component of the vector of the jumps of displacements in the fluid phase that is normal to the surface makes a nonzero contribution to the displacement field.

We illustrate the application of formulas (2.6)–(2.9) by deriving the forces equivalent to the three sources in the isotropic Biot model defined by the following jumps of displacements across the plane $\xi_3 = 0$:

$$[u_3^{(2)}] = F(t) \delta(\xi_1) \delta(\xi_2), \quad [u_1^{(2)}] = [u_2^{(2)}] = [u_3^{(1)}] = 0; \quad (3.5)$$

$$[u_3^{(1)}] = F(t) \delta(\xi_1) \delta(\xi_2), \quad [u_1^{(2)}] = [u_2^{(2)}] = [u_3^{(2)}] = 0; \quad (3.6)$$

$$[u_1^{(2)}] = F(t) \delta(\xi_1) \delta(\xi_2), \quad [u_2^{(2)}] = [u_3^{(2)}] = [u_3^{(1)}] = 0. \quad (3.7)$$

For each of the sources (3.5)–(3.7), the additional conditions

$$[\sigma] = [\sigma_{13}] = [\sigma_{23}] = [\sigma_{33}] = 0 \quad (3.8)$$

are assumed to be satisfied.

Applying formulas (2.8) and (2.9), we find the forces corresponding to the first source:

$$\begin{aligned} \vec{f}^{(1)}(\vec{\eta}, \tau) &= -MF(\tau) \text{grad}\{\delta(\eta_1) \delta(\eta_2) \delta(\eta_3)\}, \\ f_1^{(2)}(\vec{\eta}, \tau) &= -AF(\tau) \delta'(\eta_1) \delta(\eta_2) \delta(\eta_3), \\ f_2^{(2)}(\vec{\eta}, \tau) &= -AF(\tau) \delta(\eta_1) \delta'(\eta_2) \delta(\eta_3), \\ f_3^{(2)}(\vec{\eta}, \tau) &= -(A + 2L)F(\tau) \delta(\eta_1) \delta(\eta_2) \delta'(\eta_3). \end{aligned} \quad (3.9)$$

Here, $A = A_{33}^{11}$, $A + 2L = A_{33}^{33}$, $L = A_{13}^{13}$, and $M = Q_{33}$ are the elastic coefficients of the isotropic Biot model. Thus, the source given as a jump of $u_3^{(2)}$ is equivalent to the superposition of (a) a center of pressure in the solid phase with the coefficient A ; (b) a center of pressure in the fluid phase with the coefficient M , and (c) a momentless couple of oppositely directed forces acting along the x_3 axis in the skeleton with the coefficient $2L$.

Similarly, the forces equivalent to the second source are as follows:

$$\left. \begin{aligned} \vec{f}^{(1)}(\vec{\eta}, \tau) \\ \vec{f}^{(2)}(\vec{\eta}, \tau) \end{aligned} \right\} = - \left\{ \begin{array}{c} R \\ M \end{array} \right\} \text{grad}\{\delta(\eta_1) \delta(\eta_2) \delta(\eta_3)\}. \quad (3.10)$$

Here, as above, $M = Q_{33}$ and R are the elastic moduli occurring in Hooke's law (1.1). Thus, the source (3.6) is equivalent to the superposition of (a) a center of pressure in the solid phase with the coefficient M and (b) a center of pressure in the fluid phase with the coefficient R .

For the third source, (3.7), the equivalent body forces are given by

$$\vec{f}^{(1)} = 0; \quad f_2^{(2)} = 0, \quad f_1^{(2)} = -LF(\tau)\delta(\eta_1)\delta(\eta_2)\delta'(\eta_3), \quad f_3^{(2)} = -LF(\tau)\delta'(\eta_1)\delta(\eta_2)\delta(\eta_3), \quad (3.11)$$

where $L = A_{13}^{13}$. Thus, a point jump of tangential displacements in the solid phase is equivalent to the superposition of two moment couples of unit forces in the skeleton. The forces in the first couple act along the x_3 axis and their arm is directed along the x_1 axis, whereas the forces in the second couple act along the x_1 axis and their arm is directed along the x_3 axis. By analogy with the case of an elastic medium [2], such a combination can be called a double momentless couple in the skeleton.

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REFERENCES

1. G. I. Petrashen, L. A. Molotkov, and P. V. Krauklis, *Waves in Layered Isotropic Elastic Media* [in Russian], Leningrad (1982).
2. K. Aki and P. Richards, *Quantitative Seismology: Theory and Methods*, W. H. Freeman (1980).
3. G. Bonnet, "Basic singular solutions for a poroelastic medium in the dynamic range," *J. Acoust. Soc. Am.*, **82**, 1758-1762 (1987).
4. L. A. Molotkov and A. V. Bakulin, "Sources of the center of dilatation and center of compression types in the Biot model," *Zap. Nauchn. Semin. POMI*, **230**, 196-213 (1995).
5. M. A. Biot, "Mechanics of deformation and acoustic propagation in porous media," *J. Appl. Phys.*, **33**, 1482-1498 (1962).
6. L. A. Molotkov and A. V. Bakulin, "The effective model of a stratified solid-fluid medium as a special case of the Biot model," *Zap. Nauchn. Semin. POMI*, **230**, 172-195 (1995).
7. G. I. Petrashen, *Foundations of the Mathematical Theory of Propagation of Elastic Waves* [in Russian], Vol. 17, Leningrad Univ. Press (1978).