Theory of seismic phase analysis using circular statistics

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Summary

Seismic phase is critical in imaging, inversion, and coherence measurements. However, noise contamination complicates its analysis, making reliable estimation challenging. We propose to leverage circular statistics, particularly the von Mises distribution, to provide a more robust framework for characterizing phase variations in seismic signals. Computational experiments with synthetic data demonstrate the effectiveness of this approach.

Introduction

Seismic phase plays a critical role in imaging, inversion, and coherence analysis. However, noise contamination complicates its estimation, creating challenges in processing and interpretation. Because of the phase periodicity, traditional linear statistical methods struggle to analyze noisy phase distributions accurately.

Accurate phase estimation is crucial in both post-stack and pre-stack applications (van der Baan and Fomel, 2009; Fomel and van der Baan, 2014; Greer et al, 2019; Holt and Lubrano, 2020; Bakulin et al., 2022, 2023). In pre-stack data, phase stability is essential for identifying reflectors, enhancing coherence, and ensuring accurate inversion and imaging. Despite the phase importance, noise in phase remains poorly understood, limiting practical data-driven assessment and conditioning methods.

In this study, we propose to apply circular statistics, specifically the von Mises distribution, to better characterize phase variations in seismic signals. Figure 1A shows time-windowed post-stack seismic field data around the target reflector, with phase distributions at 15 Hz and 45 Hz (Figures 1B and 1C, respectively). At first glance, these patterns appear multimodal or incoherent. However, deeper analysis reveals systematic phase variations that could help identify reflectors in seismic data with low signal-to-noise ratios (SNRs). Circular statistics provide an effective tool for analyzing wrapped phases, characterizing phase distributions in a data-driven manner, and revealing key signal characteristics.

The circular nature of phases

When a seismic signal is transformed into the frequency domain, it is represented as a complex number, with phase derived from the argument of this complex-valued signal. Since phase is inherently periodic, it is confined within the interval $[-\pi, \pi]$. The wrapped phase remains continuous within the interval $[-\pi, \pi]$, but contains abrupt jumps at the boundaries. These apparent discontinuities arise because phase values outside this range are mapped back into it by adding or subtracting multiples of 2π (Figure 2).

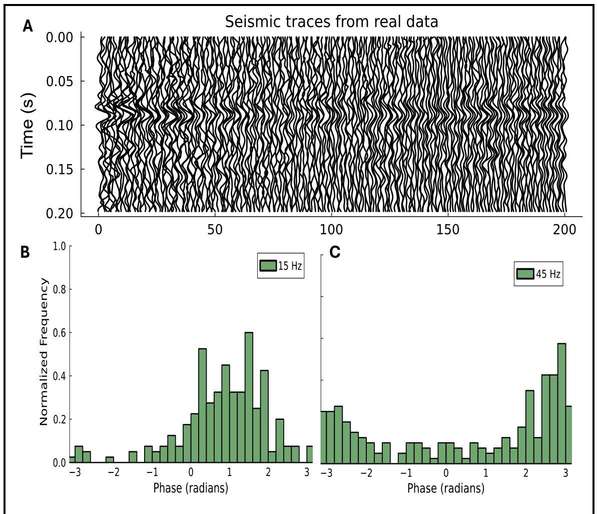

Figure 1: (A) Time-windowed data (Poland 2D Vibroseis Line) around the target reflector extracted from post-stack field data; (B, C) Distributions of wrapped phases from the real data in Figure 1A at different frequencies: (B) 15 Hz, (C) 45 Hz.

In signal processing, the phase is often unwrapped to analyze phase changes without discontinuities, especially in physical applications. While advanced phase unwrapping algorithms exist for interferometry (Ghiglia et al., 1994; Zebkar et al., 1998), they often fail in noisy regions and typically lack uncertainty estimates for derived parameters. Hence, direct analysis of the wrapped phase using circular statistics can be more practical and reliable than phase unwrapping.

von Mises Distribution

Initially introduced by von Mises (1918) as an alternative to the normal distribution for circular data, this distribution has been widely used in directional statistics (Mardia and Jupp, 2009) and radar interferogram modeling without phase unwrapping (Feigl and Clifford, 2009).

The von Mises distribution resembles a normal distribution but is adapted for circular data with truncated tails. Its probability density function is:

$$f(\theta;\bar{\theta},\kappa) = \frac{1}{2\pi I_o(\kappa)} \exp\left[\kappa\, cos(\theta - \bar{\theta})\right], \qquad (1)$$

where $I_0(x)$ is the modified Bessel function of the first kind, $\bar{\theta}$ represents the mean direction, and κ is the concentration parameter. The distribution is symmetric around the mean, which, in seismic applications, corresponds to a clean signal phase before noise or distortions.

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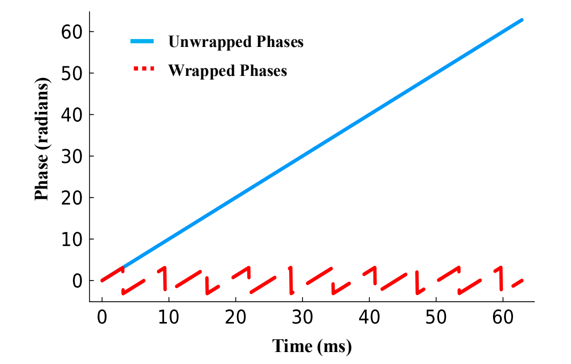

Figure 2. : Comparison of unwrapped and wrapped phases as a function of time. The unwrapped phase (blue solid line) exhibits a continuous linear increase, demonstrating the natural progression of the phase over time. In contrast, the wrapped phase (red dashed line) is confined to a fixed interval [-π, π]. This illustrates the periodic nature of wrapped phases, commonly encountered in signal processing and circular data analysis.

Impact of Noise on Phase Analysis

Seismic phase is inherently circular, making noise contamination challenging to analyze with traditional linear methods. While phase unwrapping is theoretically possible, it is often impractical due to noise and data limitations. Instead, noise directly influences the wrapped phase, shaping its statistical distribution in a way better suited to circular statistical models. The von Mises distribution provides a natural alternative, allowing robust phase characterization while preserving phase periodicity.

Noise in seismic data arises from additive distortions and phase perturbations due to scattering and statics (Goodman, 2007; Rohatgi et al., 2024). While an idealized "clean" signal phase exists, real-world data contain phase noise that affects coherence and interpretation. Because the wrapped phase is constrained within [−π, π], traditional Gaussian-based approaches fail to capture its behavior, making circular models like the von Mises distribution more appropriate.

The von Mises distribution describes phase variations through the concentration parameter κ quantifying clustering around the mean direction. As κ increases, phase values become more concentrated, approaching a normal distribution. At lower κ values, the distribution broadens, and at κ = 0, it becomes uniform over [−π, π]. Figure 3 illustrates von Mises distributions for different mean values and concentration parameters. While histograms in Figures 3A and 3B appear distorted in a linear representation, Figures 3C and 3D show that the symmetry is preserved in polar coordinates. This highlights the importance of circular representations in phase analysis. By leveraging the von Mises framework, we quantify phase variability without phase unwrapping, providing a robust method for seismic phase estimation. This approach enhances phase conditioning and noise characterization, particularly for identifying reflectors and mitigating phase distortions.

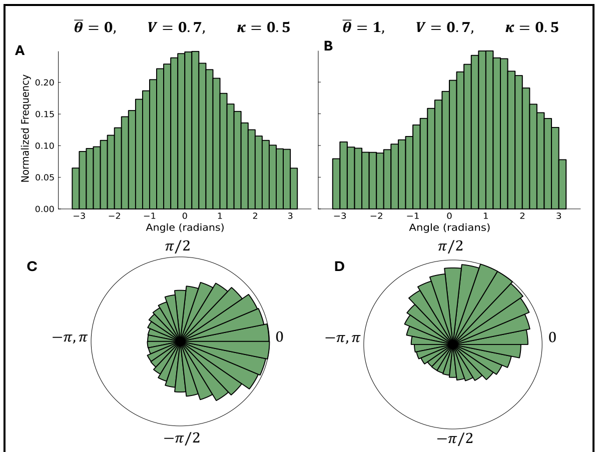

Figure 3: Histograms and polar plots of synthetic phase data sampled from von Mises distributions. (A) and (B) show histograms of 100,000 phase values drawn from von Mises distributions with parameters ($\bar{\theta} = 0$, V = 0.7, κ = 0.5) and ($\bar{\theta} = 1$, V = 0.7, κ = 0.5), respectively. (C) and (D) display the corresponding polar histograms, illustrating the circular symmetry of the distributions.

Circular Mean

While phase distributions shown in Figure 3A and 3B may appear irregular in linear space, the circular mean provides a robust method for determining the central tendency of wrapped data, avoiding discontinuities inherent in conventional averaging.

The circular mean is computed for N circular samples $(\theta_1, \theta_2, \ldots \theta_N)$ by converting phase angles into unit vectors in a two-dimensional plane. The average cosine and sine components are then used to determine the resultant vector:

$$C = \sum_{i=1}^{n} \cos\theta_i, \qquad S = \sum_{i=1}^{n} \sin\theta_i, \tag{2}$$

$$R^2 = C^2 + S^2 \; (R \geq 0). \tag{3}$$

The mean direction $\bar{\theta}$ of the vector resultant is given by

$$\bar{\theta} = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right), & S > 0, C > 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + \pi, & C < 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + 2\pi. & S < 0, C > 0 \end{cases} \tag{4}$$

The mean resultant length $\overline{R}$, defined as $\overline{R} = R/N$ measures dispersion around the mean direction and ranges between 0 and 1. A low $\overline{R}$ (Figure 4A) indicates a wide spread of phase values. A high $\overline{R}$ (Figure 4B) signifies strong phase alignment, which is helpful in assessing signal coherence. Since the circular mean accounts for the periodic nature of phase data, it provides a more reliable measure of central tendency than traditional linear methods.

Circular Variance

Circular variance measures the spread of angular data around the mean direction. Circular variance is defined as

$$V = 1 - \bar{R}\,. \tag{5}$$

It is bounded in the interval [0,1]. Additionally, phase variability can be further quantified using the concentration parameter κ, which we discuss next.

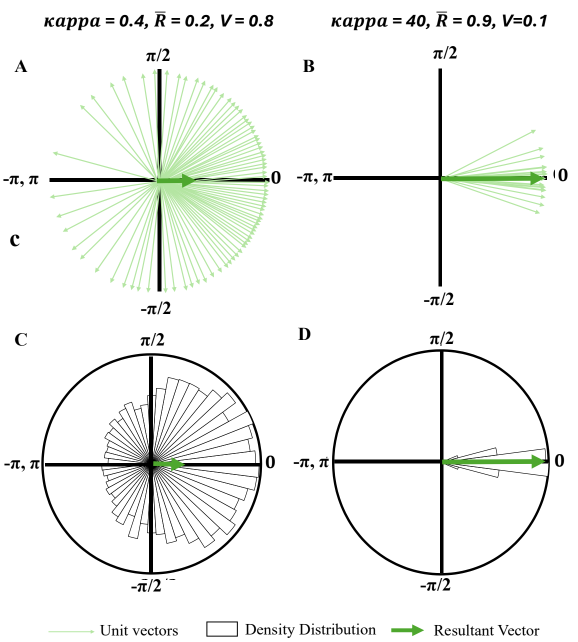

Figure 4: Polar representation of phase distributions for different kappa values in the von Mises distribution. Panels A and C (left) correspond to a low kappa (0.4), showing high phase variability, while panels B and D (right) illustrate a high kappa (40), indicating a concentrated phase distribution. The top row depicts individual phase vectors, while the bottom row visualizes the corresponding polar histograms of phase angles.

Concentration parameter

The concentration parameter κ in the von Mises distribution measures how tightly phase angles cluster around the mean direction, serving as an analog to variance in the normal distribution. A low κ (high V) indicates a nearly uniform distribution, where phase values are widely dispersed, reflecting high variability and low coherence (Figures 4A, 4C). As κ increases, the distribution becomes more concentrated, with phase values clustering more tightly around the mean direction. When κ is very large (low V), the distribution forms a sharp peak, indicating near-perfect phase alignment (Figures 4B, 4D).

Instead of performing a computational fit of the von Mises distribution, κ can be estimated directly using the mean resultant length $\overline{R}$. Empirical relations from Fisher (1993) allow an approximation of κ based on $\overline{R}$:

$$\kappa = \begin{cases} 2\overline{R} + R^3 + \dfrac{5R^5}{6}, & \overline{R} < 0.53 \\ -0.4 + 1.39\overline{R} + \dfrac{0.43}{1-\overline{R}}, & 0.53 \le \overline{R} < 0.53 \\ \dfrac{1}{3R - 4R^2 + R^3}, & \overline{R} \ge 0.85 \end{cases} \tag{6}$$

This approach simplifies the estimation of κ from phase data, avoiding complex numerical fitting while providing a reliable measure of phase coherence.

Seismic Phase Analysis

To validate the effectiveness of circular statistics in seismic phase estimation, we apply this approach to a controlled synthetic example. The application of circular statistics in seismic phase analysis is demonstrated using a simple case of a horizontal reflector in the presence of additive white noise. While the phase of the signal at each frequency remains constant across all traces, the noise component exhibits a uniformly distributed phase. The interaction of signal and noise results in a phase distribution that appears random but follows a von Mises distribution. The center of this distribution aligns with the signal phase while the noise magnitude determines its spread.

Figure 5A presents the seismic traces perturbed with the additive Gaussian noise with an SNR of -10 dB. The signal vector in the frequency domain at m^{th} frequency is expressed as

$$X_m = |X_m|\, e^{i\theta_m}, \tag{7}$$

where θ_m is the phase. Mapping this to a unit circle, the phase unit vector is given by:

$$\frac{X_m}{|X_m|} = e^{i\theta_m}. \tag{8}$$

For n number of traces, the phase unit vectors are $[\theta_m^1, \theta_m^2, \ldots \theta_m^n]$. The circular mean at each frequency is computed as

$$\overline{\theta}[\mathrm{m}] = tan^{-1}\left(\frac{\sum_{n=1}^{n} \sin(\phi_m^n)}{\sum_{n=1}^{n} \cos(\phi_m^n)}\right). \tag{9}$$

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Phase statistics can be analyzed within localized ensembles of traces. Figures 5C and 5D depict the phase distributions on a linear scale, where the distribution at 60 Hz exhibits a bimodal character. However, this apparent bimodality is an artifact of linear representation. When the same data is visualized on a circular scale (Figures 5E and 5F), a more intuitive pattern emerges, emphasizing the necessity of circular statistics in seismic phase analysis.

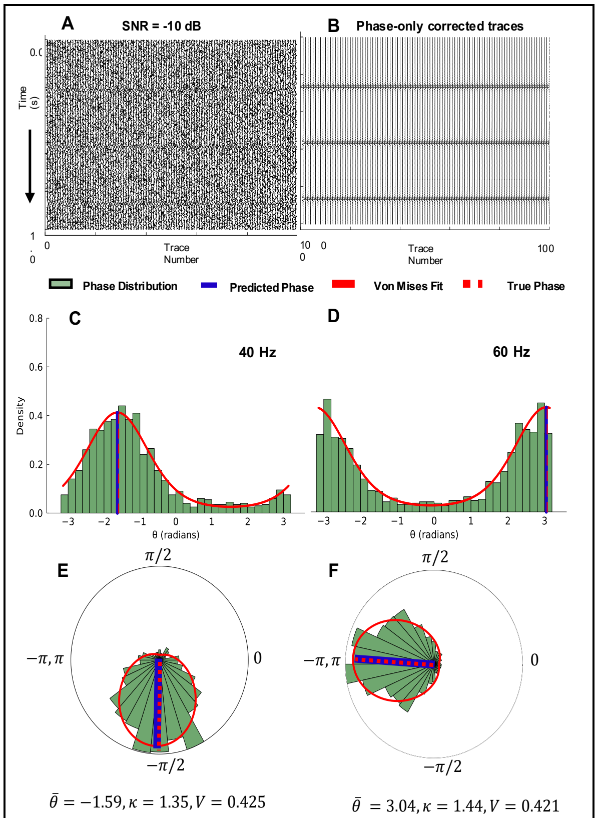

Figure 5: Phase correction of seismic traces, (A) Noisy seismic traces with SNR = -10 dB. (B) Phase-only corrected seismic traces. (C-D) Phase distributions at 40 Hz and 60 Hz, respectively, with histogram representation (green), von Mises fit (red), predicted phase (blue), and true phase (dotted red). (E-F) Circular phase distributions for 40 Hz and 60 Hz, respectively, show estimated mean direction, circular variance, and concentration parameter.

By estimating the circular mean within a local ensemble, the true phase of the signal can be recovered. In Figure 5, the red dashed lines represent this estimated signal phase. Substituting the circular mean phase in the frequency domain allows for phase correction before transforming the signals back into the time domain. The resulting traces, corrected locally through circular mean substitution, are shown in Figure 5B. In this study, amplitude variations are assumed to remain unchanged.

The analysis further demonstrates that while the circular mean adapts to the signal phase, the concentration parameter κ remains constant, consistent with the characteristics of white noise. Since white noise has a flat power spectral density, its κ does not vary with frequency. In contrast, colored noise exhibits frequency-dependent κ reflecting variations in phase coherence across different frequencies.

The concentration parameter κ and circular variance V serve as valuable measures of phase coherence, providing insight into the consistency of seismic phases. As κ increases, phase values become more tightly clustered around the mean direction, indicating greater coherence and lower noise levels. Bakulin et al. (2024) established a numerical relationship between κ and the standard deviation of the unwrapped phase, which could provide additional insights into phase stability. However, unwrapped phase estimation is impractical for seismic data. Circular statistics offer a more efficient and data-driven approach for analyzing phase variability in seismic signals.

By quantifying the degree of coherence in seismic phase distributions, κ or V guides phase correction strategies. In datasets affected by noise, κ and V help differentiate between regions with stable phase alignment and those dominated by random variability, enabling targeted signal enhancement. Conceptually, κ is analogous to the signal-to-noise ratio (SNR): just as SNR measures signal fidelity, κ and V quantify phase perturbations across different frequencies.

Conclusions

We demonstrate that treating the noisy seismic phase as a random variable provides valuable insights for multi-channel data, both post-stack and pre-stack. Most random noise, including additive and multiplicative speckle noise, leads to symmetric phase distributions centered around the true signal phase, with the spread determined by noise magnitude. When noise levels are high, phase unwrapping may be necessary for meaningful analysis in a linear space.

However, by applying the von Mises distribution and circular statistics, we effectively analyze noisy phases in the wrapped domain, accurate signal phase recovery through the circular mean, and the characterization of phase distortions via the concentration parameter or circular variance.

While we present results for a simple local ensemble, this methodology is straightforward to extend to fully data-driven characterization of entire post-stack and pre-stack volumes. This approach facilitates signal phase recovery and enables the characterization of phase distributions, supporting frequency-dependent denoising strategies.

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